



MODULE 1 NOTES

CALCULUS AND LINEAR ALGEBRA		Semester	1
Course Code	1BMATS101	CIE Marks	50
Teaching Hours/Week (L:T:P: S)	3:2:0:0	SEE Marks	50
Total Hours of Pedagogy	40Hours Theory + 20Hours Tutorial	Total Marks	100
Credits	4	Exam Hours	3 hrs
Examination type (SEE)	Theory		
Course Outcomes			
CO1: Apply the concepts of multivariable calculus and vector calculus to compute derivatives, optimize functions, and analyze vector fields for applications in computer science engineering.			

Module-1: Calculus	(8Hours Theory + 4Hours Tutorial)
Partial differentiation, total derivative, differentiation of composite functions, Jacobian, Statement of Taylor's and Maclaurin's series expansion for two variables. Maxima and minima for the function of two variables.	

15/09/25

Module - 1 CALCULUS.

Calculus & Linear Algebra

Topic - Partial Differentiation.

Let $u = f(x, y)$ be a function in 2 variables x and y . The derivative of ' u ' w.r.t ' x ' treating ' y ' as a constant is called partial derivative of ' u ' w.r.t ' x ' and is denoted by $\frac{\partial u}{\partial x}$ or u_x .

Similarly partial derivative of ' u ' w.r.t ' y ' treating ' x ' as a constant is denoted by $\frac{\partial u}{\partial y}$ or u_y .

eg: $u = x^3 + y^3 + xy^2 + 3y$

$$\frac{\partial u}{\partial x} \text{ or } u_x = 3x^2 + 0 + y^2(1) + 0$$

$$\frac{\partial u}{\partial y} \text{ or } u_y = 0 + 3y^2 + (2y)x + 3$$

NOTE: $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

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* Problems:

1) If $u = x^3 - 3xy^2 + x + e^x \cos y + 1$.

Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

$$u = x^3 - 3xy^2 + x + e^x \cos y + 1.$$

Solⁿ: $\frac{\partial u}{\partial x} = 3x^2 - 3y^2(1) + 1 + \cos y (e^x)$

$$\frac{\partial^2 u}{\partial x^2} = 6x - 0 + 0 + \cos y e^x.$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + e^x \cos y \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = 0 - 3x(2y) + 0 + e^x (-\sin y).$$

$$\frac{\partial^2 u}{\partial y^2} = -6x(1) - e^x \cos y \quad \text{--- (2)}$$

$$\text{(1) + (2)}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + e^x \cos y - 6x - e^x \cos y.$$

$$\underline{\underline{= 0}}$$

2) ~~***~~ Given, $z = \sin(ax+y) + \cos(ax-y)$, prove that

$$\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}.$$

Solⁿ: $\frac{\partial z}{\partial x} = \cos(ax+y)(a) - \sin(ax-y)(a)$

$$\frac{\partial z}{\partial x^2} = -\sin(ax+y)(a^2) - \cos(ax-y)(a^2)$$

$$\frac{\partial^2 z}{\partial x^2} = a^2 (-\sin(ax+y) - \cos(ax-y)) \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial y} = \cos(ax+y) - \sin(ax-y)(-1)$$

$$= \cos(ax+y) + \sin(ax-y)$$

$$\frac{\partial^2 z}{\partial y^2} = -\sin(ax+y) + \cos(ax-y)(-1)$$

$$\frac{\partial^2 z}{\partial y^2} = -\sin(ax+y) - \cos(ax-y) \quad \text{--- (1)}$$

② in ① gives

$$\boxed{\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}}$$

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3) Verify

$$u_{xy} = u_{yx}, \text{ for } u = \sin^{-1}\left(\frac{y}{x}\right)$$

Solⁿ: P.D w.r.t 'y'.

$$u_y = \frac{1}{\sqrt{1 - \left(\frac{y}{x}\right)^2}} \times \frac{1}{x} \quad (1)$$

$$u_y = \frac{1}{\sqrt{\frac{x^2 - y^2}{x^2}}} \times \frac{1}{x} \quad (\text{LCM})$$

$$u_y = \frac{x}{\sqrt{x^2 - y^2}} \times \frac{1}{x}$$

$$u_y = \frac{1}{\sqrt{x^2 - y^2}} = \frac{1}{(x^2 - y^2)^{1/2}} = (x^2 - y^2)^{-1/2}$$

P.D w.r.t 'x'.

$$u_{xy} = \frac{-1}{2} (x^2 - y^2)^{-1/2 - 1} (-2x)$$

$$\boxed{u_{xy} = \frac{-x}{(x^2 - y^2)^{3/2}}}$$

$$u = \sin^{-1}\left(\frac{y}{x}\right)$$

P.D w.r.t 'x'

$$u_x = \frac{1}{\sqrt{1 - \left(\frac{y}{x}\right)^2}} \times y \times \frac{-1}{x^2} \quad \left(\frac{1}{x} = \frac{-1}{x^2}\right)$$

$$u_x = \frac{1}{\sqrt{\frac{x^2 - y^2}{x^2}}} \times y \times \frac{-1}{x^2} \quad (\text{LCM})$$

$$u_x = \frac{\cancel{x}}{\sqrt{x^2 - y^2}} \times \frac{-y}{\cancel{x^2}}$$

$$u_x = \frac{-y}{x\sqrt{x^2 - y^2}}$$

P.D w.r.t 'y'.

(u_x)

$\left(\frac{u}{v} \text{ rule}\right)$

$$u_{yx} = \frac{-1}{x} \left\{ \frac{\sqrt{x^2 - y^2} (1) - y \times \frac{1}{x\sqrt{x^2 - y^2}} \times -xy}{(\sqrt{x^2 - y^2})^2} \right\} \rightarrow \text{LCM}$$

$$u_{yx} = \frac{-1}{x(x^2 - y^2)} \left\{ \frac{x^2 - y^2 + y^2}{\sqrt{x^2 - y^2}} \right\}$$

$$u_{yx} = \frac{-x}{(x^2 - y^2)^{3/2}} \quad (2)$$

$$\therefore u_{xy} = u_{yx}$$

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4) If $u = e^{ax+by} \cdot f(ax-by)$, prove that $b \frac{\partial u}{\partial x} + a \frac{\partial u}{\partial y} = 2abu$.

Solⁿ: PT: $b \frac{\partial u}{\partial x} + a \frac{\partial u}{\partial y} = 2abu$.

P.D w.r.t x

$$\frac{\partial u}{\partial x} = e^{ax+by} \cdot f'(ax-by)(a) + f(ax-by) \times e^{ax+by} \times a. \quad (\text{uv rule}).$$

$\times$ by b

$$b \frac{\partial u}{\partial x} = abe^{ax+by} [f'(ax-by) + f(ax-by)] \quad \text{--- (1) (common)}$$

P.D 'u' w.r.t 'y'.

$$\frac{\partial u}{\partial y} = e^{ax+by} \cdot f'(ax-by)(-b) + f(ax-by) \cdot e^{ax+by} \times b.$$

$\times$ by 'a'

$$a \frac{\partial u}{\partial y} = abe^{ax+by} [-f'(ax-by) + f(ax-by)] \quad \text{--- (2)}$$

① + ②.

$$b \frac{\partial u}{\partial x} + a \frac{\partial u}{\partial y} = abe^{ax+by} [f'(ax-by) + f(ax-by) - f'(ax-by) + f(ax-by)]$$

$$= ab e^{ax+by} [2f(ax-by)]$$

$$= 2ab \left[\frac{e^{ax+by}}{u} f(ax-by) \right]$$

$$= 2abu.$$

5) If $v = e^{a\theta} \cos(a \log r)$, P.T

P.T: $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} = 0.$

Solⁿ: P.D 'v' w.r.t 'r'

$$\frac{\partial v}{\partial r} = e^{a\theta} \cdot -\sin(a \log r) \times a \times \frac{1}{r}$$

$$= -ae^{a\theta} \left[\sin(a \log r) \cdot \frac{1}{r} \right]$$

$$\frac{\partial^2 v}{\partial r^2} = -ae^{a\theta} \left[\sin(a \log r) \cdot \frac{-1}{r^2} + \frac{1}{r} \cos(a \log r) \cdot \left(\frac{-a}{r} \right) \right]$$

P.D 'v' w.r.t 'θ'

$$\frac{\partial v}{\partial \theta} = \cos(a \log r) \cdot e^{a\theta} \times a$$

$$\frac{\partial^2 v}{\partial \theta^2} = \cos(a \log r) \cdot e^{a\theta} \times a^2$$

$$= a^2 e^{a\theta} \cos(a \log r)$$

Consider, each term of

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2}$$

$$\frac{\partial^2 v}{\partial r^2} =$$

$$\Rightarrow -ae^{a\theta} \left[\sin(a \log r) \cdot \frac{-1}{r^2} + \frac{1}{r} \cos(a \log r) \cdot \left(\frac{-a}{r} \right) \right]$$

$$\frac{a}{r^2} e^{a\theta} \sin(a \log r) - \frac{a^2}{r^2} e^{a\theta} \cos(a \log r).$$

$$\Rightarrow \frac{1}{r} \frac{\partial v}{\partial r} = \frac{1}{r} (-a e^{a\theta} (\sin(a \log r) \cdot \frac{1}{r}))$$

$$= -\frac{a}{r^2} e^{a\theta} (\sin(a \log r)).$$

$$\Rightarrow \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} = \frac{1}{r^2} [a^2 e^{a\theta} \cos(a \log r)]$$

$$= \frac{a^2 e^{a\theta} \cos(a \log r)}{r^2}$$

$$\therefore \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} =$$

$$\Rightarrow \frac{a}{r^2} e^{a\theta} \sin(a \log r) - \frac{a^2}{r^2} e^{a\theta} \cos(a \log r) - \frac{a e^{a\theta}}{r^2}$$

$$\sin(a \log r) + \frac{a^2 e^{a\theta} \cos(a \log r)}{r^2}$$

$$= 0$$

6) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, then, P.T

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z} \text{ \& hence show that,}$$

$$\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right]^2 u = \frac{-9}{(x+y+z)^2}$$

$$\text{sol}^n: \frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} (3x^2 - 3yz) \quad (1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} (3y^2 - 3xz) \quad (2)$$

$$\frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz} (3z^2-3xy). \quad (3)$$

$$(1) + (2) + (3).$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{1}{x^3+y^3+z^3-3xyz} \overset{\text{(common)}}{(3x^2-3yz+3y^2-3xz+3z^2-3xy)}.$$

$$= \frac{1}{x^3+y^3+z^3-3xyz} \{ (3)(x^2-yz+y^2-xz+z^2-xy) \}$$

$$= \frac{1}{x^3+y^3+z^3-3xyz} \left\{ \begin{aligned} &a^3+b^3+c^3-3abc \\ &= (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \end{aligned} \right.$$

$$= \frac{3}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} (x^2-yz+y^2-zx+z^2-xy)$$

$$= \frac{3}{x+y+z} \quad (1)$$

Now,

Consider,

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 u = \underline{9u}$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left\{ \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) u \right\}$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right)$$

from ①

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) \left[\frac{3}{x+y+z} \right]$$

$$= 3 \left[\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right] \left[\frac{1}{x+y+z} \right]$$

$$= 3 \left\{ \frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2} \right\}$$

$$= 3 \left\{ \frac{-3}{(x+y+z)^2} \right\}$$

$$= \frac{-9}{(x+y+z)^2}$$

* 7) If $z(x+y) = x^2 + y^2$, show that

$$\left[\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right]^2 = 4 \left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right]$$

Soln: $z = \frac{x^2 + y^2}{x+y}$

($\frac{u}{v}$ rule)

$$\frac{\partial z}{\partial x} = \frac{x+y(2x) - (x^2+y^2)(1)}{(x+y)^2} \quad \text{①}$$

$$\frac{\partial z}{\partial y} = \frac{x+y(2y) - (x^2+y^2)(1)}{(x+y)^2} \quad \text{②}$$

$$(0 - 0)^2$$

$$\left[\frac{(x+y)(2x) - (x^2+y^2)}{(x+y)^2} - \frac{[(x+y)(2y) - (x^2+y^2)]}{(x+y)^2} \right]^2$$

$$= \left[\frac{(x+y)(2x) - \cancel{(x^2+y^2)} - (x+y)(2y) + \cancel{(x^2+y^2)}}{(x+y)^2} \right]^2$$

$$= \left[\frac{\cancel{x+y}(2x-2y)}{(x+y)^2} \right]^2$$

$$= \left[\frac{2x-2y}{x+y} \right]^2$$

$$= \left[\frac{2(x-y)}{x+y} \right]^2$$

$$= \frac{4(x-y)^2}{(x+y)^2} \quad \text{--- (3')}$$

Consider

$$4 \left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} \right]$$

$$4 \left[1 - \frac{(x+y)(2x) - (x^2+y^2)}{(x+y)^2} - \frac{(x+y)(2y) - (x^2+y^2)}{(x+y)^2} \right]$$

$$4 \left[\frac{(x+y)^2 - \cancel{2x^2} - 2xy - \cancel{x^2} - \cancel{y^2} - 2xy - \cancel{2y^2} - \cancel{x^2} - \cancel{y^2}}{(x+y)^2} \right]$$

$$4 \left[\frac{(x+y)^2 - 2xy - 2xy}{(x+y)^2} \right]$$

$$4 \left[\frac{x^2+y^2 + \cancel{2xy} - \cancel{2xy} - 2xy}{(x+y)^2} \right]$$

$$4 \frac{(x-y)^2}{(x+y)^2} \text{ (4)}$$

$$\textcircled{3} = \textcircled{4} \Rightarrow \text{L.H.S} = \text{R.H.S.}$$

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8) ~~***~~ $u = \log(\tan x + \tan y + \tan z)$. Show that

$$\sin^2 x \frac{\partial u}{\partial x} + \sin^2 y \frac{\partial u}{\partial y} + \sin^2 z \frac{\partial u}{\partial z} = 2.$$

Solⁿ:

$$\frac{\partial u}{\partial x} = \frac{1}{(\tan x + \tan y + \tan z)} \times \sec^2 x$$

$$\frac{\partial u}{\partial y} = \frac{1}{\tan x + \tan y + \tan z} \times \sec^2 y$$

$$\frac{\partial u}{\partial z} = \frac{1}{\tan x + \tan y + \tan z} \times \sec^2 z$$

$$\frac{\sin^2 x \times \sec^2 x}{\tan x + \tan y + \tan z} + \frac{\sin^2 y \times \sec^2 y}{\tan x + \tan y + \tan z} + \frac{\sin^2 z \times \sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{1}{\tan x + \tan y + \tan z} \left[\sin^2 x \times \sec^2 x + \sin^2 y \times \sec^2 y + \sin^2 z \times \sec^2 z \right]$$

$$= \frac{1}{\tan x + \tan y + \tan z} \left[\frac{1}{\cos^2 x} x 2 \sin x \cos x + \frac{1}{\cos^2 y} x 2 \sin y \cos y + \frac{1}{\cos^2 z} x 2 \sin z \cos z \right]$$

$$= \frac{1}{\tan x + \tan y + \tan z} (2 \tan x + 2 \tan y + 2 \tan z)$$

$$= \frac{1}{\cancel{\tan x + \tan y + \tan z}} (2) (\cancel{\tan x + \tan y + \tan z})$$

$$= \underline{\underline{2}}$$

9) If $u = x^y$, show that

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$$

Solⁿ: $u = x^y$

P.D w.r.t 'y'

$$\frac{\partial u}{\partial y} = x^y \cdot \log x$$

P.D w.r.t 'x'

$$\frac{\partial^2 u}{\partial x \partial y} = x^y \frac{1}{x} + \log x [y \cdot x^{y-1}] \quad (x^n)$$

$$\frac{\partial^2 u}{\partial x \partial y} = x^{y-1} + \log x (y \cdot x^{y-1})$$

P.D w.r.t 'x'

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x} \left[x^{y-1} + \log x (y \cdot x^{y-1}) \right] \quad \text{--- (1)}$$

Consider, $u = x^y$

P.D w.r.t x .

$$\frac{\partial u}{\partial x} = y x^{y-1}$$

P.D w.r.t y

$$\frac{\partial^2 u}{\partial y \partial x} = y [x^{y-1} \log x] + x^{y-1} (1)$$

P.D w.r.t (x)

$$\frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial}{\partial x} \left\{ x^{y-1} + y x^{y-1} \log x \right\} \quad (2)$$

From (1) & (2)

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$$

* Total differentiation.

If $u = f(x, y)$, then the total differential @ the exact differential of u is defined as,

$$\rightarrow du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy.$$

① Find the total differential of
 $u = x^3 + xy^2 + x^2y + y^3.$

$$\text{Side } du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy$$

$$du = (3x^2 + y^2 + 2xy)dx + (2xy + x^2 + 3y^2)dy.$$

$$(2) x = r \sin \theta \cos \phi.$$

$$x \rightarrow f(r, \theta, \phi).$$

$$\text{Sol}^n: dx = \frac{\partial x}{\partial r} \cdot dr + \frac{\partial x}{\partial \theta} \cdot d\theta + \frac{\partial x}{\partial \phi} \cdot d\phi.$$

$$dx = \sin \theta \cdot \cos \phi \cdot dr + r \cos \phi (-\cos \theta) \cdot d\theta + r \sin \theta (-\sin \phi) d\phi \\ = (\sin \theta \cdot \cos \phi) dr + (-r \cos \phi \cos \theta) d\theta - (r \sin \theta \sin \phi) d\phi.$$

⊛ Total derivative of composite function in one variable.

If $u = f(x, y)$, where x & y are functions of the independent variable t then u is said to be composite function of single variable t .

If $u = f(x, y)$, where $x = x(t)$ and $y = y(t)$, then the total derivative is given by,

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

① Find the total derivative of $z = xy^2 + x^2y$, given $x = at$, $y = 2at$. and also verify by direct substitution.

$$\text{Sol}^n: z = f(x, y) \rightarrow g(t)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\left| \begin{array}{l} z = xy^2 + x^2y \\ x = at \\ y = 2at \end{array} \right.$$

$$\frac{dz}{dt} = (y^2 + 2xy) \cdot (a) + (2yx + x^2) \cdot (2a)$$

$$\frac{dz}{dt} = (4a^2t^2 + 4a^2t^2)a + (4a^2t^2 + a^2t^2)(2a)$$

$$\frac{dz}{dt} = 8a^2t^2(a) + 5a^2t^2(2a)$$

$$\frac{dz}{dt} = 8a^3t^2 + 10a^3t^2$$

$$\frac{dz}{dt} = \underline{\underline{18a^3t^2}}$$

direct subsⁿ

$$Z = xy^2 + x^2y$$

$$= (at)(4a^2t^2) + (a^2t^2)(2at)$$

$$= 4a^3t^3 + 2a^3t^3$$

$$Z = 6a^3t^3$$

$$\frac{dz}{dt} = 6a^3(3t^2)$$

$$= \underline{\underline{18a^3t^2}}$$

★ ***

② Find $\frac{du}{dt}$, when $u = x^3y^2 + x^2y^3$, with $x = at^2$,

$$y = 2at$$

solⁿ: $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$

$$\frac{du}{dt} = (y^2(3x^2 + y^3(2x)) 2at + (x^3 \cdot 2y + x^2 \cdot 3y^2) (2a)$$

$$\frac{du}{dt} = (3x^2y^2 + 2xy^3) 2at + (2yx^3 + 3x^2y^2) (2a)$$

$$\frac{du}{dt} = (3a^2t^4 \cdot 4a^2t^2 + 2at^2 \cdot 8a^3t^3)(2at) + (a^3t^6 \cdot 2 \times 2at + a^2t^4 \times 3 \times 4a^2t^2)(2a)$$

$$= (12a^4t^6 + 16a^4t^5)(2at) + (4a^4t^7 + 12a^4t^6)(2a)$$

$$= 24a^5t^7 + 32a^5t^6 + 8a^5t^7 + 24a^5t^6$$

$$\frac{du}{dt} = 56a^5t^6 + 32a^5t^7$$

 (3) Find total derivative of u w.r.t t

$$u = \tan^{-1}\left(\frac{x}{y}\right), \quad x = e^t - e^{-t}$$

$$y = e^t + e^{-t}$$

$$\text{sol}^n: \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{du}{dt} = \left[\frac{1}{1 + \frac{x^2}{y^2}} \times \frac{1}{y} (e^t + e^{-t}) \right] + \left[\frac{1}{1 + \left(\frac{x}{y}\right)^2} \times \frac{-x}{y^2} \times (e^t - e^{-t}) \right]$$

$$= \frac{y^2}{x^2 + y^2} \times \frac{1}{y} (e^t + e^{-t}) - \frac{y^2}{y^2 + x^2} \times \frac{x}{y^2} \times (e^t - e^{-t})$$

$$= \frac{y(e^t + e^{-t})}{x^2 + y^2} - \frac{x(e^t - e^{-t})}{y^2 + x^2}$$

$$\begin{aligned}
 & \frac{(e^t - e^{-t})^2 + (e^t + e^{-t})^2}{(e^t - e^{-t})^2 + (e^t + e^{-t})^2} \\
 &= \frac{(e^t + e^{-t})^2 - (e^t - e^{-t})^2}{(e^t - e^{-t})^2 + (e^t + e^{-t})^2} \\
 &= \frac{(e^{2t} + e^{-2t} + 2e^t e^{-t}) - (e^{2t} + e^{-2t} - 2e^t e^{-t})}{(e^{2t} + e^{-2t} - 2e^t e^{-t}) + (e^{2t} + e^{-2t} + 2e^t e^{-t})} \\
 &= \frac{(e^{2t} + e^{-2t} + 2) - (e^{2t} + e^{-2t} - 2)}{(e^{2t} + e^{-2t} - 2) + (e^{2t} + e^{-2t} + 2)} \\
 &= \frac{4}{2e^{2t} + 2e^{-2t}} \\
 &= \frac{2}{e^{2t} + e^{-2t}}
 \end{aligned}$$

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4) If $u = x^2 - y^2$, where $x = e^t \cos t$ & $y = e^t \sin t$.
 show that, $\frac{du}{dt} = 2e^{2t} [\cos 2t - \sin 2t]$.

Soln: $\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$

$$\frac{du}{dt} = 2x(e^t \cos t + \sin t e^t) + (-2y)(e^t \cos t + \sin t e^t)$$

$$\begin{aligned}
&= +2x(e^t \sin t + e^t \cos t) - 2y(e^t \cos t + e^t \sin t) \\
&= +2(e^t \cos t)(e^t \sin t + e^t \cos t) - 2(e^t \sin t)(e^t \cos t + e^t \sin t) \\
&= +2e^t \cos t (e^t \sin t + e^t \cos t) - 2e^t \sin t (e^t \cos t + e^t \sin t) \\
&= +2e^t \cos t (e^t) (\sin t + \cos t) - 2e^t \sin t (e^t) (\cos t + \sin t) \\
&= +2e^{2t} \cos t (\sin t + \cos t) - 2e^{2t} \sin t (\cos t + \sin t) \\
&= -2e^{2t} \cos t \sin t + 2e^{2t} \cos^2 t - 2e^{2t} \sin t \cos t - 2e^{2t} \sin^2 t \\
&= 2e^{2t} (\cos^2 t - \cos t \sin t - \sin t \cos t - \sin^2 t) \\
&= 2e^{2t} (\cos 2t - 2 \cos t \sin t)
\end{aligned}$$

$$\frac{du}{dt} = 2e^{2t} (\cos 2t - \sin 2t)$$

*
 $\cos 2x = \cos^2 x - \sin^2 x$
 $2 \sin x \cos x = \sin 2x$

5) If $u = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$,

$$S.T \frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$$

Solⁿ: $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$

$$\frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}} \times 3 + \frac{1}{\sqrt{1-(x-y)^2}} (-1) \times 12t^2$$

$$\frac{du}{dt} = \frac{3}{\sqrt{1-(3t-4t^3)^2}} = \frac{12t^2}{\sqrt{1-(3t-4t^3)^2}}$$

$$\frac{du}{dt} = \frac{3-12t^2}{\sqrt{1-(3t-4t^3)^2}}$$

$$\frac{du}{dt} = \frac{3(1-4t^2)}{\sqrt{1-9t^2-16t^6+24t^4}}$$

$$\frac{du}{dt} = \frac{3}{\sqrt{\frac{1-9t^2+24t^4-16t^6}{(1-4t^2)^2}}}$$

$$= \frac{3}{\sqrt{\frac{1-9t^2+24t^4-16t^6}{1-8t^2+16t^4}}}$$

$$= \frac{3}{\sqrt{1-t^2}}$$

taking inside sq. root

$$\begin{array}{r} 1-t^2 \\ 1-9t^2+24t^4-16t^6 \\ \hline 1-8t^2+16t^4 \\ (-) \quad (+) \quad (-) \\ \hline -t^2+8t^4-16t^6 \\ (-) \quad (+) \quad (-) \quad (+) \\ \hline 0 \end{array}$$

25/09/25

① Total derivative of composite functions in two variables.

If $u = f(x, y)$, where $x = x(r, s)$, $y = y(r, s)$ then the total derivative of 'u' wrt r and s is given by.

$$u = f(x, y) \rightarrow (r, s)$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

① If $z = f(x, y)$ and $x = e^u \sin v$, $y = e^u \cos v$, prove that $\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u} \left\{ \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \right\}$.

$$\text{sol}^n: \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot (e^u \sin v) + \frac{\partial z}{\partial y} \cdot (e^u \cos v) \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} (e^u \cos v) + \frac{\partial z}{\partial y} (-e^u \sin v) \quad \text{--- (2)}$$

squaring & adding ① & ②.

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = \left[\frac{\partial z}{\partial x} e^u \sin v + \frac{\partial z}{\partial y} e^u \cos v \right]^2 + \left[\frac{\partial z}{\partial x} e^u \cos v - \frac{\partial z}{\partial y} e^u \sin v \right]^2$$

$\hookrightarrow (a+b)^2 + (a-b)^2$

$$\begin{aligned} \left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 &= \left(\frac{\partial z}{\partial x}\right)^2 e^{2u} \sin^2 v + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u} \cos^2 v + \\ &\quad 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} e^{2u} \sin v \cos v + \left(\frac{\partial z}{\partial x}\right)^2 e^{2u} \cos^2 v + \\ &\quad \left(\frac{\partial z}{\partial y}\right)^2 e^{2u} \sin^2 v - 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} e^{2u} \cos v \sin v \\ &= \left(\frac{\partial z}{\partial x}\right)^2 e^{2u} (\sin^2 v + \cos^2 v) + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u} (\cos^2 v + \sin^2 v) \end{aligned}$$

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = e^{2u} \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \right]$$

2) If $u = e^{ax+by} \cdot f(ax-by)$, P.T $b \frac{\partial u}{\partial x} + a \frac{\partial u}{\partial y} = 2abu$,

by using the concept of composite functions.

Solⁿ: $u = e^{ax+by} \cdot f(ax-by)$

Let, $x = ax+by$

$y = ax-by$

$$u = e^x \cdot f(s)$$

$$u \rightarrow f(x, s) \rightarrow g(x, y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y}$$

$$\begin{aligned} \frac{\partial u}{\partial x} &= (e^x f(s)) (a) + (e^x f'(s)) (a) \\ &= a e^x f(s) + a e^x f'(s) \end{aligned}$$

xly by b

$$b \frac{\partial u}{\partial x} = a b e^x f(s) + a b e^x f'(s) = a b e^x [f(s) + f'(s)] \quad \text{--- (1)}$$

$$\begin{aligned} \frac{\partial u}{\partial y} &= (f(s) \cdot e^x) (b) + e^x f'(s) \cdot (-b) \\ &= b e^x f(s) - b e^x f'(s) \end{aligned}$$

xly by a

$$\begin{aligned} a \frac{\partial u}{\partial y} &= a b e^x f(s) - a b e^x f'(s) \\ &= a b e^x [f(s) - f'(s)] \quad \text{--- (2)} \end{aligned}$$

$$\textcircled{1} + \textcircled{2}$$

$$\begin{aligned} b \frac{\partial u}{\partial x} + a \frac{\partial u}{\partial y} &= a b e^x 2f(s) \\ &= 2 a b e^x f(s) \end{aligned}$$

$$\boxed{\frac{b \partial u}{\partial x} + \frac{a \partial u}{\partial y} = 2 a b u}$$

3) ^{***} If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$. Prove that $xu_x + yu_y + zu_z = 0$.

Solⁿ: Let $\left(p = \frac{x}{y}, q = \frac{y}{z}, r = \frac{z}{x}\right)$

$$u = f(p, q, r) \rightarrow g(x, y, z)$$

$$u_x \text{ (1) } \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$u_x = \frac{\partial u}{\partial p} \left(\frac{1}{y}\right) + \frac{\partial u}{\partial q} (0) + \frac{\partial u}{\partial r} \left(\frac{-z}{x^2}\right) \quad \text{--- (1)}$$

$$u_y \text{ (2) } \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} \quad \text{--- (2)}$$

$$u_y = \frac{\partial u}{\partial p} \left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial q} \left(\frac{1}{z}\right) + \frac{\partial u}{\partial r} (0) \quad \text{--- (2)}$$

$$u_z \text{ (3) } \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z}$$

$$u_z = \frac{\partial u}{\partial p} (0) + \frac{\partial u}{\partial q} \left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial r} \left(\frac{1}{x}\right) \quad \text{--- (3)}$$

Consider,

$$xu_x + yu_y + zu_z = \frac{x}{y} \frac{\partial u}{\partial p} - \frac{z}{x} \frac{\partial u}{\partial r} - \frac{x}{y} \frac{\partial u}{\partial p} + \frac{y}{x} \frac{\partial u}{\partial q} - \frac{y}{z} \frac{\partial u}{\partial q} + \frac{z}{x} \frac{\partial u}{\partial r}$$

$$= 0$$

4) ~~***~~ If $u = f(x-y, y-z, z-x)$

Show that : $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Solⁿ: $p = x-y, q = y-z, r = z-x$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}(1) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial y}$$

$$= \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q}(1) + \frac{\partial u}{\partial r}(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial z}$$

$$= \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial r}(1)$$

Considering LHS

$$\frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} - \frac{\partial u}{\partial q} + \frac{\partial u}{\partial r} = 0$$

= RHS

5) If $u = f(2x-3y, 3y-4z, 4z-2x)$

Find the value of $\frac{1}{2} \frac{\partial u}{\partial x} + \frac{1}{3} \frac{\partial u}{\partial y} + \frac{1}{4} \frac{\partial u}{\partial z}$

$p = 2x - 3y, q = 3y - 4z, r = 4z - 2x$

Solⁿ: $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial x}$
 $= \frac{\partial u}{\partial p} (2) + \frac{\partial u}{\partial q} (0) + \frac{\partial u}{\partial r} (-2)$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial y}$
 $= \frac{\partial u}{\partial p} (-3) + \frac{\partial u}{\partial q} (3) + 0$

$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \frac{\partial r}{\partial z}$
 $= \frac{\partial u}{\partial p} (0) + \frac{\partial u}{\partial q} (-4) + \frac{\partial u}{\partial r} (4)$

Consider L.H.S =

$\frac{1}{2} \left(2 \frac{\partial u}{\partial p} - 2 \frac{\partial u}{\partial r} \right) + \frac{1}{3} \left(-3 \frac{\partial u}{\partial p} + 3 \frac{\partial u}{\partial q} \right) + \frac{1}{4} \left(-4 \frac{\partial u}{\partial q} + 4 \frac{\partial u}{\partial r} \right)$
 $= \frac{1}{2} (2) \left(\frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \right) + \frac{1}{3} (3) \left(-\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} \right) + \frac{1}{4} (4) \left(-\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r} \right)$
 $= \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} - \frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}$
 $= 0$

Q30) If $u = f\left(\frac{x}{z}, \frac{y}{z}\right)$, P.T. $xu_x + yu_y + zu_z = 0$.

Solⁿ: $p = \frac{x}{z}, \quad q = \frac{y}{z}$

$$u_x = \frac{\partial u}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial x}$$

$$= \frac{\partial u}{\partial p} \left(\frac{1}{z}\right) + \frac{\partial u}{\partial q} (0)$$

$$= \frac{1}{z} \frac{\partial u}{\partial p}$$

$$u_y = \frac{\partial u}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial y}$$

$$= \frac{\partial u}{\partial p} (0) + \frac{\partial u}{\partial q} \left(\frac{1}{z}\right)$$

$$= \frac{1}{z} \frac{\partial u}{\partial q}$$

||| u_z

$$u_z = \frac{\partial u}{\partial p} \left(\frac{-x}{z^2}\right) + \frac{\partial u}{\partial q} \left(\frac{-y}{z^2}\right)$$

Considering LHS,

$$x \left(\frac{1}{z}\right) \frac{\partial u}{\partial p} + y \left(\frac{1}{z}\right) \frac{\partial u}{\partial q} + z \left[\left(\frac{-x}{z^2}\right) \frac{\partial u}{\partial p} + \left(\frac{-y}{z^2}\right) \frac{\partial u}{\partial q}\right]$$

$$= \frac{x}{z} \frac{\partial u}{\partial p} + \frac{y}{z} \frac{\partial u}{\partial q} + z \left[\left(\frac{-x}{z^2}\right) \frac{\partial u}{\partial p} - \frac{y}{z^2} \left(\frac{\partial u}{\partial q}\right)\right]$$

$$= \frac{x}{z} \frac{\partial u}{\partial p} + \frac{y}{z} \frac{\partial u}{\partial q} - \frac{x}{z} \frac{\partial u}{\partial p} - \frac{y}{z} \frac{\partial u}{\partial q}$$

$$= 0$$

06/10/25

7) $z = f(x, y)$, $x = e^u + e^{-v}$, $y = e^{-u} - e^v$, PT

$$x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

Solⁿ: $z \rightarrow f(x, y) \rightarrow g(u, v)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} (e^u) + \frac{\partial z}{\partial y} (-e^{-u}) \quad (1)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} (-e^{-v}) + \frac{\partial z}{\partial y} (-e^v) \quad (2)$$

$$(1) - (2)$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} e^u - \frac{\partial z}{\partial y} e^{-u} + \frac{\partial z}{\partial x} e^{-v} + \frac{\partial z}{\partial y} e^v$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} (e^u + e^{-v}) + \frac{\partial z}{\partial y} (-e^{-u} + e^v)$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} (e^u + e^{-v}) - \frac{\partial z}{\partial y} (e^{-u} - e^v)$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$$

$$= \underline{\underline{\text{LHS}}}$$

8) If $z = f(x, y)$, $x = r \cos \theta$, $y = r \sin \theta$,
 show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2$.

Soln: $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r}$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} (\cos \theta) + \frac{\partial z}{\partial y} (\sin \theta)$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} (-r \sin \theta) + \frac{\partial z}{\partial y} (r \cos \theta)$$

Consider RHS

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2$$

$$= \left(\frac{\partial z}{\partial x} \cos \theta + \frac{\partial z}{\partial y} \sin \theta\right)^2 + \frac{1}{r^2} \left(-\frac{\partial z}{\partial x} r \sin \theta + \frac{\partial z}{\partial y} r \cos \theta\right)^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 \cos^2 \theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2 \theta + 2 \frac{\partial z}{\partial x} \cos \theta \frac{\partial z}{\partial y} \sin \theta + \frac{1}{r^2}$$

$$\left(\left(\frac{\partial z}{\partial y}\right)^2 r^2 \cos^2 \theta + \left(\frac{\partial z}{\partial x}\right)^2 r^2 \sin^2 \theta - 2 \frac{\partial z}{\partial x} r \sin \theta \frac{\partial z}{\partial y} r \cos \theta\right)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 \cos^2 \theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2 \theta + \left(\frac{\partial z}{\partial y}\right)^2 \cos^2 \theta + \left(\frac{\partial z}{\partial x}\right)^2 \sin^2 \theta$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 (\cos^2 \theta + \sin^2 \theta) + \left(\frac{\partial z}{\partial y}\right)^2 (\sin^2 \theta + \cos^2 \theta)$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

LHS

* Jacobians:

Let u & v be functions of two independent variables x & y . The Jacobian 'J' of u & v w.r.t. x & y is given by;

$$J \begin{bmatrix} u & v \\ x & y \end{bmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial(u, v)}{\partial(x, y)}$$

Similarly J of u, v, w w.r.t. x, y, z

$$J \begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

① Find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$, $u = x^2 + y^2 + z^2$
 $v = xy + yz + zx$
 $w = x + y + z$

Solⁿ:

$$J \begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} + & - & + \\ 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2x((x+z) - (y+x)) - 2y((y+z) - (y+x)) + 2z((y+z) - (x+z))$$

$$= \cancel{2xz} - \cancel{2xy} - \cancel{2yz} + \cancel{2xy} + \cancel{2xz} - \cancel{xzx}$$

$$= 0$$

② *** Find,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)}, \quad \begin{aligned} u &= x+y+z \\ v &= y+z \\ w &= z \end{aligned}$$

solⁿ:

$$J \left[\begin{matrix} u & v & w \\ x & y & z \end{matrix} \right] = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} + & - & + \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1(1-0) - 1(1-0) + 1(0-0)$$

$$= 1$$

3) If $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$.

ST: $J \left[\frac{x, y, z}{r, \theta, \phi} \right] = r^2 \sin \theta$.

Solⁿ: $J \left[\frac{x, y, z}{r, \theta, \phi} \right] = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$

$$= \begin{vmatrix} \sin \theta \cos \phi & r \cos \phi \cos \theta & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \sin \phi \cos \theta & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix}$$

$$= \sin \theta \cos \phi (0 + r^2 \sin^2 \theta \cos \phi) - r \cos \phi \cos \theta (0 - r \sin \theta \cos \theta \cos \phi) - r \sin \theta \sin \phi (-r \sin^2 \theta \sin \phi - r \sin \phi \cos^2 \theta)$$

$$= r^2 \sin^3 \theta \cos^2 \phi + r^2 \cos^2 \phi \cos^2 \theta \sin \theta + r^2 \sin^3 \theta \sin^2 \phi + r^2 \sin \theta \sin^2 \phi \cos^2 \theta$$

$$= r^2 (\sin^3 \theta \cos^2 \phi + \cos^2 \phi \cos^2 \theta \sin \theta + \sin^3 \theta \sin^2 \phi + \sin \theta \sin^2 \phi \cos^2 \theta)$$

$$= r^2 \sin \theta (\sin^2 \theta \cos^2 \phi + \cos^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta \sin^2 \phi)$$

$$= r^2 \sin \theta (\sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + \cos^2 \theta (\cos^2 \phi + \sin^2 \phi))$$

$$= r^2 \sin \theta (\sin^2 \theta (1) + \cos^2 \theta (1))$$

$$= r^2 \sin \theta (1)$$

4) If $u = x + y + z$, $v = y + z$, $uvw = z$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = ?$$

Solⁿ:
$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$x + y + z = u$$

$$y + z = v$$

$$(-) \quad (-) \quad (-)$$

$$\boxed{x = u - v}$$

$$\boxed{z = uvw}$$

$$y + z = v$$

$$\boxed{y = v - uvw}$$

$$J = \begin{vmatrix} 1 & -1 & 0 \\ -vw & 1 - uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$\Rightarrow R_3 \rightarrow R_3 + R_2$$

$$= \begin{vmatrix} 1 & -1 & 0 \\ -vw & 1 - uw & -uv \\ 0 & 1 & 0 \end{vmatrix} = 1 \begin{vmatrix} 0 & uv \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} + 1 \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}$$

$$\boxed{J = uv}$$

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~~11-11-11~~
 5) If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$

Show that $\frac{\partial(u, v, w)}{\partial(x, y, z)} = 4$

$$\text{Sol}^n: \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2 y^2 z^2} \begin{vmatrix} -yz & zx & yx \\ zy & -zx & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{yz \cdot zx \cdot yx}{x^2 y^2 z^2} \begin{vmatrix} + & - & + \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \frac{yz \cdot zx \cdot yx}{x^2 y^2 z^2} [-1(1-1) - 1(-1-1) + 1(1+1)]$$

$$= \frac{yz \cdot zx \cdot yx}{x^2 y^2 z^2} [0 - 1(-2) + 1(2)]$$

$$= \frac{yz \cdot zx \cdot yx}{x^2 y^2 z^2} (0 + 2 + 2)$$

$$= \frac{yz \cdot zx \cdot yx}{x^2 y^2 z^2} (4)$$

$$= 4 = \text{RHS}$$

6) If $u = x + 3y^2 - z^3$
 $v = 4x^2 yz$
 $w = 2z^2 - xy$

Find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ at $(1, -1, 0)$

Solⁿ: $x=1, y=-1, z=0$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 6y & -3z^2 \\ xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

Put $x=1, y=-1, z=0$

$$= \begin{vmatrix} 1 & 6(-1) & 0 \\ 0 & 0 & 4(1)^2(-1) \\ -(-1) & -1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} + & - & + \\ 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= 1(0-4) + 6(0+4) + 0$$

$$= -4 + 24$$

$$= 20$$

7) If $u = x(1-y), v = xy$

Find $J \left[\frac{u,v}{x,y} \right]$

Soln:

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\Rightarrow u = x(1-y)$$

$$u = x - xy$$

$$v = xy$$

$$= \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix}$$

$$= (1-y)(x) + x(y)$$

$$= x - xy + xy$$

$$= x$$

8) If $x+y+z=u$
 $y+z=uv$
 $z=uvw$

Find $J \begin{bmatrix} x,y,z \\ u,v,w \end{bmatrix}$

$$y = uv - z$$

$$y = uv - uvw$$

$$z = uvw$$

Soln: $x+y+z=u$
 $x+uv=u$
 $x = u - uv$

$$y+z=uv$$

$$= \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-vw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$R_2 \Rightarrow R_2 + R_3$$

$$= \begin{vmatrix} 1-v & -u & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}$$

$$= 1-v(u^2v-0) + u(uv^2-0) + 0$$

$$= u^2v - \cancel{u^2v^2} + \cancel{u^2v^2}$$

$$= \frac{u^2v}{1}$$

 a) If $u = \frac{2yz}{x}$, $v = \frac{3zx}{y}$, $w = \frac{4xy}{z}$

Ans = 96

find $J \left[\frac{u, v, w}{x, y, z} \right]$

solⁿ: $\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

$= \begin{vmatrix} -\frac{2yz}{x^2} & \frac{2z}{x} & \frac{2y}{x} \\ \frac{3z}{y} & -\frac{3zx}{y^2} & \frac{3x}{y} \\ \frac{4y}{z} & \frac{4x}{z} & -\frac{4xy}{z^2} \end{vmatrix}$

$= \frac{1}{x^2 y^2 z^2} \begin{vmatrix} -2yz & 2xz & yx \\ 3yz & -3zx & 3xy \\ 4yz & 4xz & -4xy \end{vmatrix}$

$= \frac{yz \cdot xz \cdot yx}{x^2 y^2 z^2} \begin{vmatrix} + & - & + \\ -2 & 2 & 2 \\ 3 & -3 & 3 \\ 4 & 4 & -4 \end{vmatrix}$

$= -2(12 - 12) - 2(-12 - 12) + 2(12 + 12)$

$= -2(-24) + 2(24)$

$= 48 + 48$

= 96

10) If $u + v = e^x \sin y$
 $u - v = e^x \cos y$

Ans: $\frac{e^{2x}}{2}$

Find $\frac{\partial(u,v)}{\partial(x,y)}$

Solⁿ: $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

add

$\Rightarrow u + v = e^x \sin y$
 $u - v = e^x \cos y$

$2u = e^x \sin y + e^x \cos y$

$2u = e^x (\sin y + \cos y)$

$u = \frac{e^x}{2} (\sin y + \cos y)$

sub

$u + v = e^x \sin y$
 $u - v = e^x \cos y$
 $(+) (-)$

$2v = e^x (\sin y - \cos y)$

$v = \frac{e^x}{2} (\sin y - \cos y)$

$J = \begin{vmatrix} \frac{e^x}{2} (\sin y + \cos y) & \frac{e^x}{2} (\cos y - \sin y) \\ \frac{e^x}{2} (\sin y - \cos y) & \frac{e^x}{2} (\cos y + \sin y) \end{vmatrix}$

$= \frac{e^x}{2} \times \frac{e^x}{2} \begin{vmatrix} \sin y + \cos y & \cos y - \sin y \\ \sin y - \cos y & \cos y + \sin y \end{vmatrix}$

$= \frac{e^x}{2} \times \frac{e^x}{2} [(\sin y + \cos y)^2 + (\sin y - \cos y)^2]$

$= \frac{e^x}{2} \times \frac{e^x}{2} [\sin^2 y + \cos^2 y + 2 \sin y \cos y + \sin^2 y + \cos^2 y - 2 \sin y \cos y]$

$= \frac{e^x}{2} \times \frac{e^x}{2} (2 \sin^2 y + 2 \cos^2 y)$

$$= \frac{e^{2x}}{4} (2) (\sin^2 y + \cos^2 y)$$

$$\boxed{J = \frac{e^{2x}}{2}}$$

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11) If $x = a \cosh u \cos v$, $y = a \sinh u \sin v$

P.T $\frac{\partial(x, y)}{\partial(u, v)} = \frac{a^2}{2} (\cosh 2u - \cos 2v)$

Solⁿ: $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} a \sinh u \cos v & -a \cosh u \sin v \\ a \cosh u \sin v & a \sinh u \cos v \end{vmatrix}$$

$$= (a^2 \sinh^2 u \cos^2 v + a^2 \cosh^2 u \sin^2 v)$$

$$= a^2 \sinh^2 u \left[\frac{1 + \cos 2v}{2} \right] + a^2 \cosh^2 u \left[\frac{1 - \cos 2v}{2} \right]$$

$$= \frac{a^2}{2} \left[\sinh^2 u + \sinh^2 u \cos 2v + \cosh^2 u - \cosh^2 u \cos 2v \right]$$

$$= \frac{a^2}{2} \left[\sinh^2 u + \cosh^2 u + \cos 2v (\sinh^2 u - \cosh^2 u) \right]$$

$$= \frac{a^2}{2} \left[\cosh^2 u - \cos 2v (\cosh^2 u - \sinh^2 u) \right]$$

$$\begin{aligned} \because \sinh^2 u + \cosh^2 u &= \cosh 2u \\ \cosh^2 u - \sinh^2 u &= 1 \end{aligned}$$

$$\boxed{J = \frac{a^2}{2} [\cosh 2u - \cos 2v]}$$

⊛ Taylor's series expansion for functions of two variables:

$$f(x, y) = f(a, b) + [(x-a) f_x(a, b) + (y-b) f_y(a, b)] \\ + \frac{1}{2!} [(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b) \\ + (y-b)^2 f_{yy}(a, b)] + \frac{1}{3!} [(x-a)^3 f_{xxx}(a, b) + \\ 3(x-a)^2(y-b) f_{xxy}(a, b) + 3(x-a)(y-b)^2 \\ f_{xyy}(a, b) + (y-b)^3 f_{yyy}(a, b)] + \dots \rightarrow \textcircled{1}$$

The above series which expands $f(x, y)$ in infinite power series of terms containing powers of $(x-a)$ and $(y-b)$ is known as Taylor's series of $f(x, y)$ about the point (a, b) .

⊛ Maclaurin's series expansion for functions of two variables:

This is a special case of Taylor's series when the expansion is about the origin $(0, 0)$.

Thus, by replacing $a=0$ & $b=0$ in eqn $\textcircled{1}$, we get Maclaurin's series, which is given by,

$$f(x, y) = f(0, 0) + [x f_x(0, 0) + y f_y(0, 0)] + \frac{1}{2!} [x^2 f_{xx}(0, 0) \\ + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \frac{1}{3!} [x^3 f_{xxx}(0, 0) \\ + 3x^2y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] \\ + \dots$$

Thus Maclaurin's series is a series of powers of x & y .

Problems:

1) Expand $e^x \log(1+y)$ in powers of x & y upto terms of third degree.

solⁿ: $f(x, y) = e^x \log(1+y) \Rightarrow \boxed{f(0, 0) = 0}$

$$f_x(x, y) = e^x \log(1+y) \Rightarrow \boxed{f_x(0, 0) = 0}$$

$$f_y(x, y) = e^x \cdot \frac{1}{1+y} \Rightarrow \boxed{f_y(0, 0) = 1}$$

$$f_{xx}(x, y) = e^x \cdot \log(1+y) \Rightarrow \boxed{f_{xx}(0, 0) = 0}$$

$$f_{xy}(x, y) = e^x \cdot \frac{1}{1+y} \Rightarrow \boxed{f_{xy}(0, 0) = 1}$$

$$f_{yy}(x, y) = \frac{-e^x}{(1+y)^2} \Rightarrow \boxed{f_{yy}(0, 0) = -1}$$

$$f_{xxx}(x, y) = e^x \log(1+y) \Rightarrow \boxed{f_{xxx}(0, 0) = 0}$$

$$f_{xxy}(x, y) = e^x \cdot \frac{1}{1+y} \Rightarrow \boxed{f_{xxy}(0, 0) = 1}$$

$$f_{xyy}(x, y) = \frac{-e^x}{(1+y)^2} \Rightarrow \boxed{f_{xyy}(0, 0) = -1}$$

$$f_{yyy}(x, y) = \frac{2e^x}{(1+y)^3} \Rightarrow \boxed{f_{yyy}(0, 0) = 2}$$

Maclaurin's expansion of $f(x, y)$ is given by.

$$f(x, y) = f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2!} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \frac{1}{3!} [x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] + \dots$$

$$\Rightarrow e^x \log(1+y) = 0 + x(1) + y(1) + \frac{1}{2} [x^2(0) + 2xy(1) + y^2(-1)] + \frac{1}{6} [x^3(0) + 3x^2 y(1) + 3xy^2(1) + y^3(0)] + \dots$$

$$\therefore e^x \log(1+y) = y + xy - \frac{y^2}{2} + \frac{1}{6} [3x^2 y - 3xy^2 + 2y^3] + \dots$$

$$e^x \log(1+y) = y + xy - \frac{y^2}{2} + \frac{1}{2} [x^2 y - xy^2] + \frac{1}{3} y^3 + \dots$$

is the Maclaurin's expansion upto terms of third degree.

2) Find Maclaurin's expansion of $e^{ax} \sin(by)$ upto 3rd degree terms.

$$\text{sol}^n: f(x, y) = e^{ax} \sin(by) \Rightarrow f(0, 0) = 0$$

$$f_x(x, y) = a e^{ax} \sin(by) \Rightarrow f_x(0, 0) = 0$$

$$f_y(x, y) = b e^{ax} \cos(by) \Rightarrow f_y(0, 0) = b$$

$$f_{xx}(x, y) = a^2 e^{ax} \sin(by) \Rightarrow f_{xx}(0, 0) = 0$$

$$f_{xy}(x,y) = abe^{ax} \cos(by) \Rightarrow \boxed{f_{xy}(0,0) = ab}$$

$$f_{yy}(x,y) = -b^2 e^{ax} \sin(by) \Rightarrow \boxed{f_{yy}(0,0) = 0}$$

$$f_{xxx}(x,y) = a^3 e^{ax} \sin(by) \Rightarrow \boxed{f_{xxx}(0,0) = 0}$$

$$f_{xxy}(x,y) = a^2 b e^{ax} \cos(by) \Rightarrow \boxed{f_{xxy}(0,0) = a^2 b}$$

$$f_{xyy}(x,y) = -ab^2 e^{ax} \sin(by) \Rightarrow \boxed{f_{xyy}(0,0) = 0}$$

$$f_{yyy}(x,y) = -b^3 e^{ax} \cos(by) \Rightarrow \boxed{f_{yyy}(0,0) = -b^3}$$

Maclaurin's expansion of $f(x,y)$ is given by,

$$f(x,y) = f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2!} [x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \frac{1}{3!} [x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)] + \dots$$

$$\Rightarrow e^{ax} \sin(by) = 0 + 0 + y(b) + \frac{1}{2} [0 + 2abxy + 0] + \frac{1}{6} [0 + 3a^2 bx^2 y + 0 - b^3 y^3] + \dots$$

$$\therefore e^{ax} \sin(by) = (by + abxy) + \frac{1}{6} (3a^2 bx^2 y - b^3 y^3) + \dots$$

is the Maclaurin's expansion.

3) Expand $e^x \cos y$ in powers of x & y upto terms containing third degree.

$$\underline{\text{Sol}^n}: f(x, y) = e^x \cos y \Rightarrow \boxed{f(0, 0) = 1}$$

$$f_x(x, y) = e^x \cos y \Rightarrow \boxed{f_x(0, 0) = 1}$$

$$f_y(x, y) = -e^x \sin y \Rightarrow \boxed{f_y(0, 0) = 0}$$

$$f_{xx}(x, y) = e^x \cos y \Rightarrow \boxed{f_{xx}(0, 0) = 1}$$

$$f_{xy}(x, y) = -e^x \sin y \Rightarrow \boxed{f_{xy}(0, 0) = 0}$$

$$f_{yy}(x, y) = -e^x \cos y \Rightarrow \boxed{f_{yy}(0, 0) = -1}$$

$$f_{xxx}(x, y) = e^x \cos y \Rightarrow \boxed{f_{xxx}(0, 0) = 1}$$

$$f_{xxy}(x, y) = -e^x \sin y \Rightarrow \boxed{f_{xxy}(0, 0) = 0}$$

$$f_{xyy}(x, y) = -e^x \cos y \Rightarrow \boxed{f_{xyy}(0, 0) = -1}$$

$$f_{yyy}(x, y) = e^x \sin y \Rightarrow \boxed{f_{yyy}(0, 0) = 0}$$

Maclaurin's expansion of $f(x, y)$ is given by -

$$f(x, y) = f(0, 0) + x f_x(0, 0) + y f_y(0, 0) + \frac{1}{2!} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \frac{1}{3!} [x^3 f_{xxx}(0, 0) + 3x^2 y f_{xxy}(0, 0) + 3xy^2 f_{xyy}(0, 0) + y^3 f_{yyy}(0, 0)] + \dots$$

$$\Rightarrow e^x \cos y = 1 + x(1) + 0 + \frac{1}{2} [x^2(1) + 0 + y^2(-1)] + \frac{1}{6} [x^3(1) + 0 + 3xy^2(-1) + 0] + \dots$$

$$\therefore e^x \cos y = 1 + x + \frac{1}{2} [x^2 - y^2] + \frac{1}{6} [x^3 - 3xy^2] + \dots$$

is the Maclaurin's expansion.

① Assignment.

- 1) Expand $e^x \log(1+x)$ in powers of x & y upto terms of third degree.
- 2) Find Maclaurin's series of $e^x \sin y$ upto terms of third degree.
- 3) Expand $\cos x \cos y$ in powers of x & y .

② Maxima and Minima for a function of two variables:

NOTE: For a function of one variable, $f(a)$ is maximum value of $f(x)$ if $f'(a) = 0$ & $f''(a) < 0$ and $f(a)$ is minimum value of $f(x)$ if $f'(a) = 0$ & $f''(a) > 0$.

③ Similarly for two variables:

$f(x, y)$ is said to have a min value at (a, b) if $f(a, b) < f(a+h, b+k)$, which is a nbd point.

Also $f(x, y)$ is said to have a max value at (a, b) if $f(a, b) > f(a+h, b+k)$. $f(a, b)$ is said to be an extreme value of $f(x, y)$ if it is a max value

④ a min value.

Working procedure for finding extreme values of $f(x, y)$.

- 1) Find the stationary points (x, y) such that $f_x = 0$ & $f_y = 0$.
- 2) Find, $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$. Then evaluate these at all stationary points and find $AC - B^2$.
- 3) (i) A stationary point (x_0, y_0) is a maximum point if $AC - B^2 > 0$ & $A < 0$.
 $f(x_0, y_0)$ is a max. value.
- (ii) A stationary point (x_1, y_1) is a min point if $AC - B^2 > 0$ & $A > 0$.
 $f(x_1, y_1)$ is a min value.

NOTE: $AC - B^2 < 0$, $AC - B^2 = 0$, $A = 0$, the point (x, y) is called saddle point.

- 1) Find the extreme values of the functions
 $f(x, y) = x^3 + y^3 - 3x - 12y + 20$.

Solⁿ: $f_x = 3x^2 - 3$, $f_y = 3y^2 - 12$

$$f_x = 0 \Rightarrow 3x^2 - 3 = 0 \Rightarrow x = \pm 1$$

$$f_y = 0 \Rightarrow 3y^2 - 12 = 0 \Rightarrow y = \pm 2$$

$\therefore (1, 2), (1, -2), (-1, 2), (-1, -2)$ are the stationary points.

Let $A = f_{xx} = 6x$, $B = f_{xy} = 0$, $C = f_{yy} = 6y$.

	$(1, 2)$	$(1, -2)$	$(-1, 2)$	$(-1, -2)$
$A = 6x$	$6 > 0$	6	-6	$-6 < 0$
$B = 0$	0	0	0	0
$C = 6y$	12	-12	12	-12
$AC - B^2$	$72 > 0$	$-72 < 0$	$-72 < 0$	$72 > 0$
Conclusion	<u>Min</u> point	Saddle point	Saddle point	<u>Max</u> point.

Max value of $f(x, y)$ is

$$f(-1, -2) = -1 + 3 - 8 + 24 + 20 = 38$$

Min value of $f(x, y)$ is

$$f(1, 2) = 1 + 8 - 3 - 24 + 20 = 2$$

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2) Find the max & min values of the function

$$x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x.$$

Solⁿ: $f_x = 3x^2 + 3y^2 - 30x + 72$

$$f_y = 6xy - 30y.$$

$$f_x = 0 \Rightarrow 3x^2 + 3y^2 - 30x + 72 = 0$$

$$x^2 + y^2 - 10x + 24 = 0 \rightarrow \textcircled{1}$$

$$f_y = 0 \Rightarrow y(x-5) = 0 \Rightarrow y = 0 \text{ \& } x = 5.$$

$$\text{put } y = 0 \text{ in } \textcircled{1} \Rightarrow x^2 - 10x + 24 = 0$$

$$x = 4, 6$$

$$\text{put } x = 5 \text{ in } \textcircled{1} \Rightarrow 25 + y^2 - 50 + 24 = 0$$

$$y^2 = 1 \Rightarrow y = \pm 1$$

$\therefore (4, 0), (6, 0), (5, 1) \text{ \& } (5, -1)$ are stationary points

$$A = f_{xx} = 6x - 30, B = f_{xy} = 6y, C = f_{yy} = 6x - 30.$$

	$(4, 0)$	$(6, 0)$	$(5, 1)$	$(5, -1)$
$A = 6x - 30$	$-6 < 0$	$6 > 0$	0	0
$B = 6y$	0	0	6	-6
$C = 6x - 30$	-6	6	0	0
$AC - B^2$	$36 > 0$	$36 > 0$	$-36 < 0$	$-36 < 0$
Conclusion	Max. pt	Min. pt	Saddle pt	Saddle pt.

$$\text{Max value} = f(4, 0) = 112$$

$$\text{Min value} = f(6, 0) = 108.$$

3) Examine the function $xy(a - x - y)$ for extreme values.

$$\text{Sol}^n: f(x, y) = axy - x^2y - xy^2$$

$$f_x = ay - 2xy - y^2$$

$$f_y = ax - x^2 - 2xy$$

$$f_x = 0 \Rightarrow y(a - 2x - y) = 0 \rightarrow (1)$$

$$f_y = 0 \Rightarrow x(a - x - 2y) = 0 \rightarrow (2)$$

Pairing eqns,

$$y = 0 \text{ \& } x = 0 \Rightarrow (0, 0)$$

$$y = 0 \text{ \& } a - x - 2y = 0 \Rightarrow (a, 0)$$

$$a - 2x - y = 0 \text{ \& } x = 0 \Rightarrow (0, a)$$

$$a - 2x - y = 0 \text{ \& } a - x - 2y = 0$$

$$\begin{array}{r} 2x + y = a \\ x + 2y = a \times 2 \\ \hline \end{array}$$

$$\Rightarrow \begin{array}{r} 2x + y = a \\ 2x + 4y = 2a \\ \hline (-) \quad (-) \quad (-) \quad (-) \\ -3y = -a \end{array}$$

$$\boxed{y = \frac{a}{3}}$$

$$2x + \frac{a}{3} = a$$

$$2x = \frac{2a}{3}$$

$$\boxed{x = \frac{a}{3}}$$

$$2x = a - \frac{a}{3}$$

$$2x = \frac{3a - a}{3}$$

$$2x = \frac{2a}{3}$$

$\therefore$ stationary pts are $(0, 0)$, $(a, 0)$, $(0, a)$ & $(\frac{a}{3}, \frac{a}{3})$.

Let,

$$A = f_{xx} = -2y$$

$$B = f_{xy} = a - 2x - 2y$$

$$C = f_{yy} = -2x$$

	$(0,0)$	$(a,0)$	$(0,a)$	$(a/3, a/3)$
$A = -2y$	0	0	$-2a$	$-\frac{2a}{3}$
$B = a - 2x - 2y$	a	$-a$	$-a$	$-\frac{a}{3}$
$C = -2x$	0	$-2a$	0	$-\frac{2a}{3}$
$AC - B^2$	$-a^2 < 0$	$-a^2 < 0$	$-a^2 < 0$	$\frac{a^2}{3} > 0$
Conclusion	Saddle points			depends on 'a'.

Here, $AC - B^2 > 0$, A is negative if $a > 0$ & A is +ve if $a < 0$.

Thus, the given function is max (i) min according as $a > 0$ (ii) $a < 0$.

4) Examine the function $f(x, y) = x^4 + y^4 - 2(x-y)^2$ for extreme values.

Solⁿ: $f_x = 4x^3 - 4(x-y)$, $f_y = 4y^3 + 4(x-y)$

$$x^3 - (x-y) = 0 \rightarrow (1)$$

$$x - y = x^3$$

$$\Rightarrow y^3 + (x-y) = 0 \rightarrow (2)$$

$$y^3 + x^3 = 0$$

$$(x+y)(x^2 - xy + y^2) = 0$$

$$y = -x, x^2 - 2xy + y^2 = 0$$

put $y = -x$ in (1), $x^3 - 2x = 0$
 $x(x^2 - 2) = 0$

$$x = 0, \pm\sqrt{2}$$

$$\therefore y = -x \Rightarrow y = 0, -\sqrt{2}, \sqrt{2}$$

$\therefore$ Stationary pts are $(0,0), (-\sqrt{2}, \sqrt{2}), (\sqrt{2}, -\sqrt{2})$

$$\text{Also, } x^2 - xy + y^2 = 0$$

$$x(x-y) + y^2 = 0$$

$$x(x^3) + y^2 = 0 \quad (\because \text{from } \textcircled{1} \Rightarrow x^3 = x - y)$$

$$x^4 + y^2 = 0, \text{ this doesn't have any real values}$$

Let,

$$A = f_{xx} = 12x^2 - 4$$

$$B = f_{xy} = 4$$

$$C = f_{yy} = 12y^2 - 4.$$

	$(0,0)$	$(\sqrt{2}, -\sqrt{2})$	$(-\sqrt{2}, \sqrt{2})$
$A = 12x^2 - 4$	-4	$20 > 0$	$20 > 0$
$B = 4$	4	4	4
$C = 12y^2 - 4$	-4	20	20
$AC - B^2$	$0 = 0$	$384 > 0$	$384 > 0$
Conclusion	Saddle pt	Min pt	Min pt.

$$\begin{aligned} \therefore \text{Minimum value} &= f(\sqrt{2}, -\sqrt{2}) \textcircled{2} f(-\sqrt{2}, \sqrt{2}) \\ &= 4 + 4 - 16 \\ &= -8 \end{aligned}$$

5) Find the extreme values of $f(x,y) = x^3 y^2 (1-x-y)$

Solⁿ: $f_x = -x^3y^2 + (1-x-y)(3x^2y^2)$
 $= -x^3y^2 + 3x^2y^2 - 3x^3y^2 - 3x^2y^3$
 $f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3.$

$f_y = -x^3y^2 + (1-x-y)(2x^3y)$
 $= -x^3y^2 + 2x^3y - 2x^4y - 2x^3y^2$

$f_y = 2x^3y - 2x^4y - 3x^3y^2$

$f_x = 0$

$x^2y^2[3-4x-3y] = 0$

$x=0, y=0, 4x+3y=3$

$f_y = 0$

$x^3y[2-2x-3y] = 0$

$x=0, y=0, 2x+3y=2$

$\therefore$ pair of eqns are

$x=0, y=0$

$x=0, 2x+3y=2 \Rightarrow y=2/3$

$y=0, 2x+3y=2 \Rightarrow x=1$

$4x+3y=3, x=0 \Rightarrow y=1$

$4x+3y=3, y=0 \Rightarrow x=3/4$

$4x+3y=3, 2x+3y=2 \Rightarrow x=1/2, y=1/3$

$\therefore$ stationary points are $(0,0)$ $(0, 2/3)$ $(1,0)$ $(0,1)$
 $(1/2, 1/3).$

Also, $A = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$

$B = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$

$C = f_{yy} = 2x^3 - 2x^4 - 6x^3y$

	$(0,0)$	$(0, \frac{3}{4})$	$(0, \frac{2}{3})$	$(1,0)$	$(0,1)$	$(\frac{1}{2}, \frac{1}{2})$
$A = 6xy^2 - 12x^2y^2 - 6xy^3$	0	0	0	0	0	$-\frac{1}{9} < 0$
$B = 6x^2y - 8x^3y - 9x^2y^2$	0	0	0	0	0	$\frac{-1}{12}$
$C = 2x^3 - 2x^4 - 6x^3y$	0	0	0	0	0	$\frac{-1}{8}$
$AC - B^2$	0	0	0	0	0	$\frac{1}{144} > 0$
Conclusion:	Saddle points.					<u>Max pt</u>

Max value = $f(x,y) = x^3y^2(1-x-y) = \frac{1}{432}$

6) S.T, $z(x,y) = x^3 + y^3 - 3xy + 1$ is min at $(1,1)$

Solⁿ: $Z_x = 3x^2 - 3y = 0$, $Z_y = 3y^2 - 3x = 0$

At $(1,1)$, $Z_x = 0$, $Z_y = 0$

$A = Z_{xx} = 6x$, $B = Z_{xy} = -3$, $C = Z_{yy} = 6y$

At $(1,1)$, $A = 6$, $B = -3$, $C = 6$

$AC - B^2 = 27 > 0$

At $(1,1)$

$Z_x = 0$, $Z_y = 0$, $AC - B^2 > 0$ & $A = 6 > 0$

$\Rightarrow z(x,y)$ at $(1,1)$ is min

$z(1,1) = \underline{0}$

7) Examine the function $f(x, y) = 1 + \sin(x^2 + y^2)$ for extremum.

Solⁿ: $f_x = 2x \cos(x^2 + y^2)$, $f_y = 2y \cos(x^2 + y^2)$

$$2x \cos(x^2 + y^2) = 0, \quad 2y \cos(x^2 + y^2) = 0$$

$$x = 0,$$

$$y = 0$$

$\therefore (0, 0)$ is the stationary point.

$$A = f_{xx} = -4x^2 \sin(x^2 + y^2) + 2 \cos(x^2 + y^2)$$

$$B = f_{xy} = -4xy \sin(x^2 + y^2)$$

$$C = f_{yy} = -4y^2 \sin(x^2 + y^2) + 2 \cos(x^2 + y^2)$$

$$\text{At } (0, 0), A = 2, B = 0, C = 2$$

$$AC - B^2 = 4 > 0 \quad \& \quad A > 0$$

$$\therefore \text{min value of } f(x, y) = f(0, 0) = 1$$

8) Examine the function $\sin x + \sin y + \sin(x+y)$ for extreme values.

Solⁿ: $f_x = \cos x + \cos(x+y)$, $f_y = \cos y + \cos(x+y)$

$$\cos x + \cos(x+y) = 0 \rightarrow \textcircled{1}; \quad \cos y + \cos(x+y) = 0$$

$$\cos(x+y) = -\cos x; \quad \cos(x+y) = -\cos y$$

$$-\cos x = -\cos y \Rightarrow x = y$$

Put $y = x$ in $\textcircled{1}$

$$\cos x + \cos 2x = 0$$

$$\cos x + 2\cos^2 x - 1 = 0$$

$$2\cos^2 x + \cos x - 1 = 0$$

$$2\cos^2 x + 2\cos x - \cos x - 1 = 0$$

$$2\cos x [\cos x + 1] - 1 [\cos x + 1] = 0$$

$$\cos x = \frac{1}{2}, \quad \cos x = -1$$

$$x = \frac{\pi}{3}, \quad x = +\pi$$

$\therefore \left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ & (π, π) are stationary pts.

$$A = f_{xx} = -\sin x - \sin(\pi+y)$$

$$B = f_{xy} = -\sin(\pi+y)$$

$$C = f_{yy} = -\sin y - \sin(\pi+y)$$

	$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$	(π, π)
$A = -\sin x - \sin(\pi+y)$	$-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3} < 0$	0
$B = -\sin(\pi+y)$	$-\frac{\sqrt{3}}{2}$	0
$C = -\sin y - \sin(\pi+y)$	$-\sqrt{3}$	0
$AC - B^2$	$\frac{9}{4} > 0$	$0 = 0$
Conclusion	Max. point	Saddle pt.

$$\begin{aligned} \text{Max point of } f(x,y) &= \sin \frac{\pi}{3} + \sin \frac{\pi}{3} + \sin\left(\frac{2\pi}{3}\right) \\ &= \frac{3\sqrt{3}}{2} \end{aligned}$$

9) Find the extreme value of the function,
 $f(x, y) = x^2 + y^2 + 6x - 12$.

Solⁿ: $f_x = 2x + 6$, $f_y = 2y$

$$f_x = 0 \Rightarrow 2x + 6 = 0 \Rightarrow \boxed{x = -3}$$

$$f_y = 0 \Rightarrow 2y = 0 \Rightarrow \boxed{y = 0}$$

stationary point $(-3, 0)$

$$\boxed{A = f_{xx} = 2} \quad \boxed{B = f_{xy} = 0} \quad \boxed{C = f_{yy} = 2}$$

$$AC - B^2 = 4 > 0 \quad \& \quad A = 2 > 0$$

$\therefore (-3, 0)$ is minimum point.

Min value $= f(-3, 0) = 9 + 0 - 18 - 12 = \underline{-21}$

10) Find the maximum & minimum values of the function, $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.

Solⁿ: $f_x = 3x^2 + 3y^2 - 6x = 0$

$$\Rightarrow 3(x^2 + y^2 - 2x) = 0$$

$$\Rightarrow x^2 + y^2 - 2x = 0 \rightarrow \textcircled{1}$$

$$f_y = 6xy - 6y = 0$$

$$\Rightarrow 6y(x - 1) = 0$$

$$\Rightarrow y(x - 1) = 0$$

$$\boxed{y = 0} \quad \textcircled{02} \quad \boxed{x = 1}$$

when $y = 0$, $\textcircled{1} \Rightarrow x^2 - 2x = 0$

$$\Rightarrow x(x - 2) = 0$$

$$\boxed{x = 0} \quad \textcircled{03} \quad \boxed{x = 2}$$

when $x=1$, ① $\Rightarrow 1+y^2-2=0$

$\Rightarrow y^2=1$

$y=+1, y=-1$

Stationary points : $(0,0), (2,0), (1,1), (1,-1)$.

$A = f_{xx} = 6x - 6, B = f_{xy} = 6y$

$C = f_{yy} = 6x - 6$

	$(0,0)$	$(2,0)$	$(1,1)$	$(1,-1)$
$A = 6x - 6$	$-6 < 0$	$6 > 0$	0	0
$B = 6y$	0	0	6	-6
$C = 6x - 6$	-6	6	0	0
$AC - B^2$	$36 > 0$	$36 > 0$	$-36 < 0$	$-36 < 0$
Conclusion	Max. point	Min. point	Saddle points.	

Max. value $= f(0,0) = 0 + 0 + 0 + 0 + 4 = 4$.

Min. value $= f(2,0) = 8 + 0 - 12 + 0 + 4 = 0$.

11) Find the extreme values of the function $xy(1-x-y)$.

Solⁿ: $f(x,y) = xy - x^2y - xy^2$

$f_x = y - 2xy - y^2, f_y = x - x^2 - 2xy$

$f_x = 0 \Rightarrow y(1-2x-y) = 0 \rightarrow \textcircled{1}$

$f_y = 0 \Rightarrow x(1-x-2y) = 0 \rightarrow \textcircled{2}$

Pairing eq'ns,

$$\boxed{y=0} \text{ \& } \boxed{x=0} \Rightarrow (0,0)$$

$$\boxed{y=0} \text{ \& } 1-x-2y=0 \Rightarrow 1-x=0 \Rightarrow \boxed{x=1} \Rightarrow (1,0)$$

$$1-2x-y=0 \text{ \& } \boxed{x=0} \Rightarrow 1-y=0 \Rightarrow \boxed{y=1} \Rightarrow (0,1)$$

$$1-2x-y=0 \text{ \& } 1-x-2y=0$$

$$\begin{array}{rcl} 2x+y=1 & \Rightarrow & 2x+y=1 \\ x+2y=1 \times 2 & \Rightarrow & 2x+4y=2 \\ \hline & & (-) \quad (-) \quad (-) \\ & & -3y=-1 \Rightarrow \boxed{y=\frac{1}{3}} \end{array}$$

$$x+2y=1$$

$$x + \frac{2}{3} = 1 \Rightarrow \boxed{x=\frac{1}{3}} \Rightarrow \left(\frac{1}{3}, \frac{1}{3}\right)$$

$\therefore$ Stationary points are $(0,0)$, $(1,0)$, $(0,1)$ & $\left(\frac{1}{3}, \frac{1}{3}\right)$.

$$A = f_{xx} = -2y$$

$$B = f_{xy} = 1-2x-2y$$

$$C = f_{yy} = -2x$$

	$(0,0)$	$(1,0)$	$(0,1)$	$\left(\frac{1}{3}, \frac{1}{3}\right)$
$A = -2y$	0	0	-2	$-\frac{2}{3} < 0$
$B = 1-2x-2y$	1	-1	-1	$\frac{1}{3}$
$C = -2x$	0	-2	0	$-\frac{2}{3}$
$AC - B^2$	$-1 < 0$	$-1 < 0$	$-1 < 0$	$\frac{1}{3} > 0$
Conclusion	$\leftarrow$ saddle points			

$$\begin{aligned}\min \text{ value} &= f\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{9} \left[1 - \frac{1}{3} - \frac{1}{3}\right] \\ &= \frac{1}{27}.\end{aligned}$$





DEPARTMENT OF MATHEMATICS

MODULE 2 NOTES

CALCULUS AND LINEAR ALGEBRA		Semester	1
Course Code	1BMATS101	CIE Marks	50
Teaching Hours/Week (L:T:P: S)	3:2:0:0	SEE Marks	50
Total Hours of Pedagogy	40Hours Theory + 20Hours Tutorial	Total Marks	100
Credits	4	Exam Hours	3 hrs
Examination type (SEE)	Theory		
Course Outcomes			
CO1: Apply the concepts of multivariable calculus and vector calculus to compute derivatives, optimize functions, and analyze vector fields for applications in computer science engineering.			

Module-2: Vector Calculus	(8Hours Theory
+ 4Hours Tutorial)	
Scalar and vector fields, Gradient, directional derivatives, divergence and curl - physical interpretation, solenoidal vector fields, irrotational vector fields and scalar potential. Introduction to polar coordinates and polar curves. Curvilinear coordinates: Scale factors, base vectors, Cylindrical polar coordinates, Spherical polar coordinates, transformation between cartesian and curvilinear systems, orthogonality.	

Vector Calculus

Scalar point function

For every point (x, y, z) in region 'R' if there corresponds scalar $\phi(x, y, z)$ then ϕ is a scalar point function.

ex: $\phi = x^2 + y^2 + z^2$

Vector point function

For every point (x, y, z) in region 'R' if there corresponds vector $\vec{A}(x, y, z)$ then $\vec{A}$ is called vector point function.

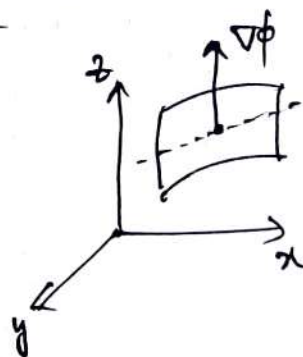
ex: $\vec{A} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

Gradient of scalar field

If $\phi(x, y, z)$ is continuously differentiable scalar point function then the gradient of ϕ ($\text{Grad } \phi$) is given by

$$\text{Grad } \phi = \nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$

Note: $\nabla \phi$ represents normal vector to the surface at given point



Divergence of vector field

If $\vec{A}(x, y, z)$ is continuously differentiable vector point function then divergence of $\vec{A}$ ($\text{div } \vec{A}$) is defined by

$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k})$$

$$\text{div } \vec{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

curl of vector field

If $\vec{A}(x, y, z)$ is continuously differentiable vector point function then curl of $\vec{A}$ is given by

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Directional derivative

If $\phi(x, y, z)$ is a scalar point function and $\vec{D}$ is a given direction directional derivative of ϕ along $\vec{D}$ is given by $\left\langle \nabla \phi \cdot \hat{n} \right\rangle$ where $\hat{n} = \frac{\vec{D}}{|\vec{D}|}$

problems.

Note: Given a vector point function $\vec{A}$.

* If $\text{curl } \vec{A} = \vec{0}$ then $\vec{A}$ is called irrotational.

* If $\text{div } \vec{A} = 0$ then $\vec{A}$ is called Solenoidal.

Q1 Find the gradient of ϕ at $(2, 1, -1)$ given

$$\phi = x^2y - 2xz + 2y^2z$$

$$\text{Sol: wbt } \nabla \phi = \left(\frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \right)$$

$$\nabla \phi = (2xy - 2z) \hat{i} + (x^2 + 4yz) \hat{j} + (-2x + 2y^2) \hat{k}$$

$$\nabla \phi \Big|_{(2, 1, -1)} = (4 - 2(-1)) \hat{i} + (4 + 4(1)(-1)) \hat{j} + (-2(2) + 2(1)) \hat{k}$$
$$\left\langle \nabla \phi = 5\hat{i} - 2\hat{k} \right\rangle$$

Q. Find the unit normal vector to the Surface

$$\phi = xy^3z^2 - 4 \text{ at } (1, -1, 2).$$

Sol. Let $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$ (Vector normal to Surface ϕ)

$$\nabla\phi = y^3z^2\hat{i} + 3xy^2z^2\hat{j} + 2xy^3z\hat{k}$$

$$\nabla\phi|_{(1, -1, 2)} = (1^3 \cdot 2^2)\hat{i} + (3(1)(-1)^2 \cdot 2)\hat{j} + (2(1)(-1)^3 \cdot 2)\hat{k}$$

$$\nabla\phi|_{(1, -1, 2)} = -4\hat{i} - 12\hat{j} + 4\hat{k}$$

unit vector normal is given by $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$\hat{n} = \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{(-4)^2 + (-12)^2 + 4^2}}$$

$$\hat{n} = \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{16 + 144 + 16}}$$

$$\hat{n} = \frac{-4\hat{i} - 12\hat{j} + 4\hat{k}}{\sqrt{176}}$$

Q. Find gradient to the Surface $\phi = 3xy^2 - y^3z^2$ also find the magnitude of $\text{grad}\phi$ at $(1, -2, -1)$

Sol. $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$$\nabla\phi = 3y^2\hat{i} + (6xy - 3y^3z^2)\hat{j} + (-2y^3z)\hat{k}$$

$$\nabla\phi|_{(1, -2, -1)} = -12\hat{i} - 9\hat{j} - 4\hat{k}$$

$$|\nabla\phi| = \sqrt{(-12)^2 + (-9)^2 + (-16)^2}$$

$$|\nabla\phi| = \sqrt{481}$$

Note: If $\vec{a}$ & $\vec{b}$ are two Vectors then angle between the vectors is given by

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

Q. Find the Angle b/w two Surfaces $x^2 + y^2 - z^2 = 4$ and $z = x^2 + y^2 - 13$ at $(2, 1, 2)$

Sol. let $\phi_1 = x^2 + y^2 - z^2 = 4$

$$\phi_2 = x^2 + y^2 - 13 - z$$

$$\nabla\phi_1 = \frac{\partial\phi_1}{\partial x}\hat{i} + \frac{\partial\phi_1}{\partial y}\hat{j} + \frac{\partial\phi_1}{\partial z}\hat{k}$$

$$\nabla\phi_2 = \frac{\partial\phi_2}{\partial x}\hat{i} + \frac{\partial\phi_2}{\partial y}\hat{j} + \frac{\partial\phi_2}{\partial z}\hat{k}$$

$$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$$

$$\nabla\phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

~~also~~

$$|\nabla\phi_1| = 4\hat{i} + 2\hat{j} - 4\hat{k}$$

(2, 1, 2)

$$|\nabla\phi_2| = 4\hat{j} + 2\hat{j} - \hat{k}$$

(2, 1, 2)

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1| |\nabla\phi_2|}$$

$$\cos\theta = \frac{(4\hat{i} + 2\hat{j} - 4\hat{k}) \cdot (4\hat{j} + 2\hat{j} - \hat{k})}{\sqrt{16+4+16} \sqrt{16+4+1}} = \frac{16+4+4}{\sqrt{36} \sqrt{21}} = \frac{24}{6\sqrt{21}}$$

$$\cos\theta = \frac{8}{\sqrt{21}} \Rightarrow \theta = \cos^{-1}\left(\frac{8}{\sqrt{21}}\right)$$

Q) Find the Angle between the directions of the normal to the surface $x^2yz=1$ at the points $(-1, 1, 1)$ & $(1, -1, -1)$ (5)

Sol: let $\phi = x^2yz - 1$

w.b.t ∇ Normal to the Surface ϕ is given by $\nabla\phi$ vector

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = 2xyz\hat{i} + x^2z\hat{j} + x^2y\hat{k}$$

$$\nabla\phi|_{(-1, 1, 1)} = -2\hat{i} + \hat{j} + \hat{k}$$

$$\nabla\phi|_{(1, -1, -1)} = 2\hat{i} - \hat{j} - \hat{k}$$

~~cos~~ w.b.t $\cos\theta = \frac{\nabla\phi|_{(-1, 1, 1)} \cdot \nabla\phi|_{(1, -1, -1)}}{|\nabla\phi|_{(-1, 1, 1)} |\nabla\phi|_{(1, -1, -1)}}$

$$\cos\theta = \frac{(-2\hat{i} + \hat{j} + \hat{k}) \cdot (2\hat{i} - \hat{j} - \hat{k})}{\sqrt{4+1+1} \sqrt{4+1+1}} = \frac{-4-1-1}{\sqrt{6}\sqrt{6}} = \frac{-6}{6}$$

$$\cos\theta = -1$$

$$\theta = \cos^{-1}(-1)$$

$$\Rightarrow \boxed{\theta = \pi}$$

Q1 Find the Lf w the Surfaces $x \log z = y^2 - 1$ and $x^2 y = z - 2$ at $(1, 1, 1)$

Solⁿ let $\phi_1 = x \log z - y^2 + 1$

$$\nabla \phi_1 = \frac{\partial \phi_1}{\partial x} \hat{i} + \frac{\partial \phi_1}{\partial y} \hat{j} + \frac{\partial \phi_1}{\partial z} \hat{k}$$

$$\nabla \phi_1 = \log z \hat{i} - 2y \hat{j} + \frac{x}{z} \hat{k}$$

$$\nabla \phi_1|_{(1,1,1)} = \log 1 \hat{i} - 2 \hat{j} + \hat{k}$$

$$\nabla \phi_1|_{(1,1,1)} = -2 \hat{j} + \hat{k}$$

$$\phi_2 = x^2 y + z - 2$$

$$\nabla \phi_2 = \frac{\partial \phi_2}{\partial x} \hat{i} + \frac{\partial \phi_2}{\partial y} \hat{j} + \frac{\partial \phi_2}{\partial z} \hat{k}$$

$$\nabla \phi_2 = 2xy \hat{i} + x^2 \hat{j} + \hat{k}$$

$$\nabla \phi_2|_{(1,1,1)} = 2 \hat{i} + \hat{j} + \hat{k}$$

$$\cos \theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| |\nabla \phi_2|} = \frac{(-2 \hat{j} + \hat{k}) \cdot (2 \hat{i} + \hat{j} + \hat{k})}{\sqrt{4+1} \sqrt{4+1+1}}$$

$$\cos \theta = \frac{0 - 2 + 1}{\sqrt{5} \sqrt{6}}$$

$$\cos \theta = \frac{-1}{\sqrt{30}} \Rightarrow \theta = \cos^{-1} \left(\frac{-1}{\sqrt{30}} \right)$$

Q2 Find the constants 'a, & b' Such that the 2 Surfaces $ax^2 - byz = (a+2)x$ and $4x^2 y + z^3 = 4$ are orthogonal at the points $(1, -1, 2)$

Solⁿ: $a = 5/2, b = 1$

hint take $\cos \theta = \pi/2$ orthogonal

Q) Find the directional derivative of $4xz^3 - 3x^2y^2z$ at $(2, -1, 2)$ along $2\hat{i} - 3\hat{j} + 6\hat{k}$ (7)

Sol: let $P.D = \nabla\phi \cdot \hat{n}$ let $\phi = 4xz^3 - 3x^2y^2z$

consider $\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$$\nabla\phi = (4z^3 - 6xy^2z)\hat{i} + (-3x^2yz)\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}$$

$$\nabla\phi|_{(2, -1, 2)} = (32 - 24)\hat{i} + 48\hat{j} + (48 - 12)\hat{k}$$

$$\nabla\phi|_{(2, -1, 2)} = 8\hat{i} + 48\hat{j} + 36\hat{k}$$

Also $\hat{n} = \frac{\vec{d}}{|\vec{d}|} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{4 + 9 + 36}} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7}$

$$P.D = \nabla\phi \cdot \hat{n} = \frac{(8\hat{i} + 48\hat{j} + 36\hat{k}) \cdot (2\hat{i} - 3\hat{j} + 6\hat{k})}{7}$$

$$\nabla\phi \cdot \hat{n} = \frac{16 - 144 + 504}{7}$$

$$\boxed{D.O = \frac{376}{7}}$$

Q) Find the directional derivative of $\phi = \frac{xz}{x^2 + y^2}$ at the point $(1, -1, 1)$ at the direction of $\hat{i} - 2\hat{j} + \hat{k}$

Sol: let $\phi = \frac{xz}{x^2 + y^2}$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = \left(\frac{(x^2+y^2)z - xz(2x)}{(x^2+y^2)^2} \right) \hat{i} + \frac{(x^2+y^2)(6) - xz(2y)}{(x^2+y^2)^2} \hat{j} + \left(\frac{x}{x^2+y^2} \right) \hat{k} \quad (2)$$

$$\nabla\phi = \left(\frac{x^2z + y^2z - 2zx^2}{(x^2+y^2)^2} \right) \hat{i} - \left(\frac{2xyx}{(x^2+y^2)^2} \right) \hat{j} + \left(\frac{x}{x^2+y^2} \right) \hat{k}$$

$$\nabla\phi = \left(\frac{y^2z - 2x^2}{(x^2+y^2)^2} \right) \hat{i} - \frac{2xyx}{(x^2+y^2)^2} \hat{j} + \left(\frac{x}{x^2+y^2} \right) \hat{k}$$

$$\nabla\phi|_{(1,-1,1)} = 0\hat{i} + \frac{1}{4}\hat{j} + \frac{1}{2}\hat{k} = \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$$

$$D \cdot D = \nabla\phi \cdot \hat{n} = \left(0\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k} \right) \cdot \frac{(\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{1+4+1}}$$

$$\nabla\phi \cdot \hat{n} = \frac{\frac{1}{2}(-1) + \frac{1}{2}(1)}{\sqrt{6}}$$

$$D \cdot D = \frac{-\frac{1}{2}}{\sqrt{6}}$$

$$\boxed{D \cdot D = \frac{-1}{2\sqrt{6}}}$$

Q. Find the divergence & curl of the vector

$xyz \hat{i} + 3x^2y \hat{j} + (xz^2 - y^2z) \hat{k}$ at the point $(2, -1, 1)$.

Sol:- let $\vec{v} = xyz \hat{i} + 3x^2y \hat{j} + (xz^2 - y^2z) \hat{k}$

Wkt $\text{div } \vec{v} = \nabla \cdot \vec{v}$

$$\text{div } \vec{v} = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot (xyz \hat{i} + 3x^2y \hat{j} + xz^2 - y^2z \hat{k})$$

$$\nabla \cdot \vec{v} = \frac{\partial}{\partial x} (xyz) + \frac{\partial}{\partial y} (3x^2y) + \frac{\partial}{\partial z} (xz^2 - y^2z)$$

$$\nabla \cdot \vec{v} = yz + 3x^2 + 2xz - y^2$$

$$|\nabla \cdot \vec{v}|_{(2, -1, 1)} = 14.$$

$$\text{curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$\nabla \times \vec{v} = \hat{i} \left[\frac{\partial}{\partial y} (xz^2 - y^2z) - \frac{\partial}{\partial z} (3x^2y) \right] - \hat{j} \left[\frac{\partial}{\partial x} (xz^2 - y^2z) - \frac{\partial}{\partial z} (xyz) \right] + \hat{k} \left[\frac{\partial}{\partial x} (3x^2y) - \frac{\partial}{\partial y} (xyz) \right]$$

$$\nabla \times \vec{v} = \hat{i} (-2yz - 0) - \hat{j} (z^2 - xy) + \hat{k} (6xy - xz)$$

$$\nabla \times \vec{v} = -2yz \hat{i} - (z^2 - xy) \hat{j} + (6xy - xz) \hat{k}$$

$$|\nabla \times \vec{v}|_{(2, -1, 1)} = 2 \hat{i} - 3 \hat{j} - 14 \hat{k}$$

Q Find the divergence & curl if $\vec{F} = \text{grad}(xy^3z^4)$ (10)

Sol let $\phi = xy^3z^4$

$$\vec{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\vec{F} = y^3z^4\hat{i} + 3xy^2z^4\hat{j} + 2xy^3z^3\hat{k}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^4 & 3xy^2z^4 & 2xy^3z^3 \end{vmatrix}$$

$$\text{curl } \vec{F} = \hat{i} \left[\frac{\partial}{\partial y}(2xy^3z^3) - \frac{\partial}{\partial z}(3xy^2z^4) \right] - \hat{j} \left[\frac{\partial}{\partial x}(2xy^3z^3) - \frac{\partial}{\partial z}(y^3z^4) \right] \\ + \hat{k} \left[\frac{\partial}{\partial x}(3xy^2z^4) - \frac{\partial}{\partial y}(y^3z^4) \right]$$

$$\text{curl } \vec{F} = (6xy^3z - 6xy^2z^4)\hat{i} - (2xy^3z - 2y^3z^4)\hat{j} + \hat{k}(3y^2z^4 - 3y^3z^4)$$

$$\text{curl } \vec{F} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\text{curl } \vec{F} = \vec{0} \quad \text{irrotational}$$

$$\text{Divergence } \vec{F} = \nabla \cdot \vec{F}$$

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot (y^3z^4\hat{i} + 3xy^2z^4\hat{j} + 2xy^3z^3\hat{k})$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(y^3z^4) + \frac{\partial}{\partial y}(3xy^2z^4) + \frac{\partial}{\partial z}(2xy^3z^3)$$

$$\nabla \cdot \vec{F} = 0 + 6xy^2z^4 + 2xy^3z^3$$

$$\nabla \cdot \vec{F} = 6xy^2z^4 + 2xy^3z^3$$

Q) Find the divergence & curl for the vector

$$\vec{F} = \nabla \phi \quad \text{where} \quad \phi = x^3 + y^3 + z^3 - 3xyz$$

Sol. Given $\vec{F} = \nabla \phi = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) (x^3 + y^3 + z^3 - 3xyz)$

$$\vec{F} = (3x^2 - 3yz) \hat{i} + (3y^2 - 3xz) \hat{j} + (3z^2 - 3xy) \hat{k}$$

Now

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$$\text{curl } \vec{F} = \hat{i} \left[\frac{\partial}{\partial y} (3z^2 - 3xy) - \frac{\partial}{\partial z} (3y^2 - 3xz) \right] - \hat{j} \left[\frac{\partial}{\partial x} (3z^2 - 3xy) - \frac{\partial}{\partial z} (3x^2 - 3yz) \right] + \hat{k} \left[\frac{\partial}{\partial x} (3y^2 - 3xz) - \frac{\partial}{\partial y} (3x^2 - 3yz) \right]$$

$$\text{curl } \vec{F} = (-3x + 3x) \hat{i} - \hat{j} (-3y + 3y) + \hat{k} (-3z + 3z)$$

$$\text{curl } \vec{F} = +0x \hat{i} + 0y \hat{j} + 0z \hat{k}$$

$$\text{curl } \vec{F} = \vec{0}$$

consider

$$\text{divergence } \vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \vec{F}$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} (3x^2 - 3yz) + \frac{\partial}{\partial y} (3y^2 - 3xz) + \frac{\partial}{\partial z} (3z^2 - 3xy)$$

$$\nabla \cdot \vec{F} = 6x + 6y + 6z$$

$$\nabla \cdot \vec{F} = 6(x + y + z)$$

Q. If $\vec{F} = \frac{x\hat{i} + y\hat{j}}{x^2 + y^2}$ then S-T $\vec{F}$ is both solenoidal & (12)

irrotational.

Sol: let $\vec{F} = \left(\frac{x}{x^2+y^2}\right)\hat{i} + \left(\frac{y}{x^2+y^2}\right)\hat{j}$

consider

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(\frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j} + 0\hat{k}\right)$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} \left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y} \left(\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial z} (0)$$

$$\nabla \cdot \vec{F} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2} + 0$$

$$\nabla \cdot \vec{F} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\nabla \cdot \vec{F} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\nabla \cdot \vec{F} = \frac{y^2 - x^2 + x^2 - y^2}{(x^2+y^2)^2} = 0$$

$$\nabla \cdot \vec{F} = 0 \Rightarrow \vec{F} \text{ is Solenoidal.}$$

consider $\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$

$$\text{curl } \vec{F} = \hat{i} \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z} \left(\frac{y}{x^2+y^2} \right) \right] - \hat{j} \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z} \left(\frac{x}{x^2+y^2} \right) \right] \\ + \hat{k} \left[\frac{\partial}{\partial x} \left(\frac{y}{x^2+y^2} \right) - \frac{\partial}{\partial y} \left(\frac{x}{x^2+y^2} \right) \right]$$

$$\text{curl } \vec{F} = \hat{i}(0-0) - \hat{j}(0-0) + \hat{k} \left(\frac{-y}{(x^2+y^2)^2} (2x) - \left(\frac{-x}{(x^2+y^2)^2} \right) (2y) \right)$$

$$\text{curl } \vec{F} = \hat{k} \left(\frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} \right)$$

$$\langle \text{curl } \vec{F} = \vec{0} \rangle \Rightarrow \vec{F} \text{ is irrotational.}$$

Q. Find the constant 'a' such that

$$\vec{F} = (x+3y)\hat{i} + (y-2x)\hat{j} + (x+az)\hat{k} \text{ is Solenoidal}$$

Q. Given $\vec{F}$ Solenoidal

$$\Rightarrow \nabla \cdot \vec{F} = 0$$

$$\left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left((x+3y)\hat{i} + (y-2x)\hat{j} + (x+az)\hat{k} \right) = 0$$

$$\frac{\partial}{\partial x} (x+3y) + \frac{\partial}{\partial y} (y-2x) + \frac{\partial}{\partial z} (x+az) = 0$$

$$1 + 1 + a = 0$$

$$2 + a = 0$$

$$\boxed{a = -2}$$

Q.1: Find $\text{curl}(\text{curl } \vec{A})$ given that $\vec{A} = xy\hat{i} + y^2z\hat{j} + z^2y\hat{k}$ 13*

Sol. $\text{curl } \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2z & z^2y \end{vmatrix}$

$$\text{curl } \vec{A} = \hat{i} \left(\frac{\partial}{\partial y}(z^2y) - \frac{\partial}{\partial z}(y^2z) \right) - \hat{j} \left(\frac{\partial}{\partial x}(z^2y) - \frac{\partial}{\partial z}(xy) \right) + \hat{k} \left(\frac{\partial}{\partial x}(y^2z) - \frac{\partial}{\partial y}(xy) \right)$$

$$\text{curl } \vec{A} = \hat{i} (z^2 - y^2) - \hat{j} (0 - 0) + \hat{k} (0 - x)$$

$$\text{curl } \vec{A} = (z^2 - y^2)\hat{i} - x\hat{k}$$

$$\text{curl}(\text{curl } \vec{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 - y^2 & 0 & -x \end{vmatrix}$$

$$\nabla \times (\nabla \times \vec{A}) = \hat{i} \left(\frac{\partial}{\partial y}(-x) - \frac{\partial}{\partial z}(0) \right) - \hat{j} \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial z}(z^2 - y^2) \right) + \hat{k} \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(z^2 - y^2) \right)$$

$$\text{curl}(\text{curl } \vec{A}) = \hat{i} (0 - 0) - \hat{j} (-1 - 2z) + \hat{k} (0 - (-2y))$$

$$\text{curl}(\text{curl } \vec{A}) = 0\hat{i} + (2z + 1)\hat{j} + 2y\hat{k}$$

Q If $\vec{F} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$ then show that $\vec{F}$ is irrotational also find scalar function ϕ such that $\vec{F} = \nabla\phi$. 44

Sol: let $\vec{F} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$

consider $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$

$$\nabla \times \vec{F} = \hat{i} \left[\frac{\partial}{\partial y} (x+y) - \frac{\partial}{\partial z} (z+x) \right] - \hat{j} \left[\frac{\partial}{\partial x} (x+y) - \frac{\partial}{\partial z} (y+z) \right] + \hat{k} \left[\frac{\partial}{\partial x} (z+x) - \frac{\partial}{\partial y} (y+z) \right]$$

$$\nabla \times \vec{F} = \hat{i}(1-1) - \hat{j}(1-1) + \hat{k}(1-1)$$

$$\nabla \times \vec{F} = \vec{0} \Rightarrow \vec{F} \text{ is irrotational.}$$

To find scalar ϕ

Given $\vec{F} = \nabla\phi$

$$\rightarrow \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$$

Equating components

$$\frac{\partial\phi}{\partial x} = y+z$$

$$\frac{\partial\phi}{\partial y} = z+x$$

$$\frac{\partial\phi}{\partial z} = x+y$$

$$\int \frac{\partial\phi}{\partial x} dx = \int (y+z) dx + f_1(x,y) \quad \left| \quad \int \frac{\partial\phi}{\partial y} dy = \int (z+x) dy + f_2(x,z) \quad \right| \quad \int \frac{\partial\phi}{\partial z} dz = \int (x+y) dz + f_3(x,y)$$

$$\textcircled{D} \Rightarrow \int \frac{\partial \phi}{\partial z} dz = \int (2y^2z + xy) dz + f_3(y, x).$$

$$\phi = \frac{2y^2z^2}{2} + xyz + f_3(x, y)$$

$$\phi = y^2z^2 + xyz + f_3(x, y) \text{ --- } \textcircled{C}$$

Choose $f_1(y, z) = y^2z^2$

$$f_2(x, z) = 0$$

$$f_3(x, y) = x^2y^2$$

$$\textcircled{A} \textcircled{B} \textcircled{C} \Rightarrow \left\langle \phi = x^2y^2 + xyz + y^2z^2 \right\rangle$$

Q Find constants 'a + b' Such that

$\vec{F} = (axy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (bxz^2 - y)\hat{k}$ is irrotational.

Also find a scalar function ϕ Such that $\vec{F} = \nabla\phi$

Q: Given $\vec{F}$ irrotational $\Rightarrow \nabla \times \vec{F} = \vec{0}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy + z^3 & 3x^2 - z & bxz^2 - y \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$\hat{i} \left(\frac{\partial}{\partial y} (bxz^2 - y) - \frac{\partial}{\partial z} (3x^2 - z) \right) - \hat{j} \left(\frac{\partial}{\partial x} (bxz^2 - y) - \frac{\partial}{\partial z} (axy + z^3) \right) + \hat{k} \left(\frac{\partial}{\partial x} (3x^2 - z) - \frac{\partial}{\partial y} (axy + z^3) \right) = \vec{0}$$

$$\hat{i} (-1 + 3z^2) - \hat{j} (bz^2 - 3xz^2) + \hat{k} (6x - ax) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

Equating components

$$-bz^2 + 3z^2 = 0 \Rightarrow -bz^2 = -3z^2 \Rightarrow \boxed{b = +3}$$

$$6x - ax = 0 \Rightarrow 6x = ax \Rightarrow \boxed{a = 6}$$

To find scalar function ϕ

$$\vec{F} = \nabla\phi$$

$$\Rightarrow (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

equating components

$$\frac{\partial\phi}{\partial x} = 6xy + z^3 \quad \text{--- ①}$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - z \quad \text{--- ②}$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 - y \quad \text{--- ③}$$

$$\textcircled{1} \Rightarrow \int \frac{\partial \phi}{\partial x} dx = \int (6xy + z^3) dx + f_1(y, z)$$

$$\phi = \frac{6x^2y}{2} + z^3x + f_1(y, z)$$

$$\phi = 3x^2y + z^3x + f_1(y, z) \text{ --- } \textcircled{A}$$

$$\textcircled{2} \Rightarrow \int \frac{\partial \phi}{\partial y} dy = \int (3x^2 - z) dy + f_2(x, z)$$

$$\phi = 3x^2y - zy + f_2(x, z) \text{ --- } \textcircled{B}$$

$$\textcircled{3} \Rightarrow \int \frac{\partial \phi}{\partial z} dz = \int (3xz^2 - y) dz + f_3(x, y)$$

$$\phi = \frac{3xz^3}{3} - yz + f_3(x, y)$$

$$\phi = xz^3 - yz + f_3(x, y) \text{ --- } \textcircled{C}$$

Choose

$$f_1(y, z) = -zy$$

$$f_2(x, z) = xz^3$$

$$f_3(x, y) = 3x^2y$$

$$\textcircled{A} \textcircled{B} \textcircled{C} \Rightarrow$$

$$\phi = \underline{\underline{3x^2y + z^3x - zy}}$$

Q. Find the value of constant a such that the vector field $\vec{F} = (axy - z^3)\hat{i} + (a-2)x^2\hat{j} + (1-a)xz^2\hat{k}$ is irrotational & hence find scalar potential ϕ such that

$$\vec{F} = \nabla \phi.$$

Sol: $a = 4 \quad \phi = 2x^2y - xz^3.$

Curvilinear co-ordinates

If (x, y, z) be the co-ordinate of the point 'p' and if 'p' is expressible in terms of new co-ordinates (u, u_2, u_3) where u, u_2, u_3 are able to expressed as functions of x, y, z then the co-ordinates ' u, u_2, u_3 ' are called curvilinear co-ordinates of point 'p'.

$$\text{i.e. } (x, y, z) \approx (u_1(x, y, z), u_2(x, y, z), u_3(x, y, z))$$

- Note (i) The Surfaces $u_1 = c_1, u_2 = c_2, u_3 = c_3$ are called co-ordinate Surfaces where c_1, c_2, c_3 are constants.
- (ii) The intersection of a pair of co-ordinate Surfaces generates a curve called as co-ordinate curve

orthogonal curvilinear co-ordinates

A System of curvilinear co-ordinates are said to be orthogonal, if the tangents drawn to the co-ordinate curves are mutually perpendicular to each other.

unit vectors, Scale factors & orthogonality

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of a point 'p' in space then

(ie $\vec{r} = \vec{r}(u, u_2, u_3)$) where u, u_2, u_3 are functions of x, y, z)

(i) Tangent vectors to the co-ordinate curves is given by $\frac{\partial \vec{r}}{\partial u_1}$, $\frac{\partial \vec{r}}{\partial u_2}$, $\frac{\partial \vec{r}}{\partial u_3}$

(ii) unit tangent vectors in the respective direction is given by

$$\hat{e}_1 = \frac{\partial \vec{r} / \partial u_1}{h_1} \quad \hat{e}_2 = \frac{\partial \vec{r} / \partial u_2}{h_2} \quad \hat{e}_3 = \frac{\partial \vec{r} / \partial u_3}{h_3}$$

$$\text{where } h_1 = \left| \frac{\partial \vec{r}}{\partial u_1} \right| \quad h_2 = \left| \frac{\partial \vec{r}}{\partial u_2} \right| \quad h_3 = \left| \frac{\partial \vec{r}}{\partial u_3} \right|$$

(iii) Here h_1 h_2 h_3 are called Scale factors.

(iv) Condition for orthogonality is given by

$$\hat{e}_1 \cdot \hat{e}_2 = 0$$

$$\hat{e}_2 \cdot \hat{e}_3 = 0$$

$$\hat{e}_3 \cdot \hat{e}_1 = 0$$

Note:

(i) If $\vec{A}$ is vector in the orthogonal curvilinear co-ordinate system, then

$$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \quad \text{Where } A_1, A_2, A_3 \text{ are functions of } u_1, u_2, u_3.$$

$$(ii) \quad \hat{e}_1 \times \hat{e}_2 = \hat{e}_3 ; \quad \hat{e}_2 \times \hat{e}_3 = \hat{e}_1 ; \quad \hat{e}_3 \times \hat{e}_1 = \hat{e}_2$$

(iii) The relation between cartesian & cylindrical polar co-ordinates is given by

$$x = \rho \cos \phi ; \quad y = \rho \sin \phi ; \quad z = z$$

(iv) The relation between cartesian & Spherical polar co-ordinates is given by

$$x = r \sin \theta \cos \phi ; \quad y = r \sin \theta \sin \phi ; \quad z = r \cos \theta$$

where $(x, y, z) \rightarrow$ cartesian system.

$(\rho, \phi, z) \rightarrow$ cylindrical system.

$(r, \theta, \phi) \rightarrow$ Spherical system.

Scale factors for cylindrical system

(4)

let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector

let the cylindrical polar coordinates (ρ, ϕ, z) is given by

$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

$$z = z$$

$$\text{then } \vec{r} = \rho \cos \phi \hat{i} + \rho \sin \phi \hat{j} + z \hat{k}$$

By definition we have

$$h_1 = \left| \frac{\partial \vec{r}}{\partial \rho} \right| = \left| \cos \phi \hat{i} + \sin \phi \hat{j} \right| = \sqrt{\cos^2 \phi + \sin^2 \phi} = 1$$

$$\begin{aligned} h_2 &= \left| \frac{\partial \vec{r}}{\partial \phi} \right| = \left| -\rho \sin \phi \hat{i} + \rho \cos \phi \hat{j} \right| = \sqrt{\rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi} \\ &= \sqrt{\rho^2 (\sin^2 \phi + \cos^2 \phi)} \end{aligned}$$

$$h_2 = \rho.$$

$$h_3 = \left| \frac{\partial \vec{r}}{\partial z} \right| = \left| 0\hat{i} + 0\hat{j} + \hat{k} \right| = \sqrt{1^2} = 1$$

$$\text{So } \left. \begin{array}{l} h_1 = 1 \\ h_2 = \rho \\ h_3 = 1 \end{array} \right\} \text{ Scale factors for cylindrical system}$$

Scale factors for Spherical System

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector

with Spherical polar coordinates (r, θ, ϕ) is given by

$$x = r \sin \theta \cos \phi ; y = r \sin \theta \sin \phi ; z = r \cos \theta$$

$$\text{Now } \vec{r} = r \sin \theta \cos \phi \hat{i} + r \sin \theta \sin \phi \hat{j} + r \cos \theta \hat{k}.$$

By definition of Scale factors we have

$$h_1 = \left| \frac{\partial \vec{r}}{\partial r} \right| = \left| \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k} \right|$$

$$= \sqrt{\sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta}$$

$$= \sqrt{\sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + \cos^2 \theta}$$

$$= \sqrt{\sin^2 \theta + \cos^2 \theta} = 1$$

$$h_2 = \left| \frac{\partial \vec{r}}{\partial \theta} \right| = \left| r \cos \theta \cos \phi \hat{i} + r \cos \theta \sin \phi \hat{j} - r \sin \theta \hat{k} \right|$$

$$= \sqrt{r^2 \cos^2 \theta \cos^2 \phi + r^2 \cos^2 \theta \sin^2 \phi + r^2 \sin^2 \theta}$$

$$= \sqrt{r^2 \cos^2 \theta (\cos^2 \phi + \sin^2 \phi) + r^2 \sin^2 \theta}$$

$$= \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta}$$

$$= \sqrt{r^2} = r$$

$$\begin{aligned}
 h_3 &= \left| \frac{\partial \vec{r}}{\partial \phi} \right| = \left| -r \sin \theta \sin \phi \hat{i} + r \sin \theta \cos \phi \hat{j} + 0 \hat{k} \right| \\
 &= \sqrt{(-r)^2 \sin^2 \theta \sin^2 \phi + r^2 \sin^2 \theta \cos^2 \phi} \\
 &= \sqrt{r^2 \sin^2 \theta (\sin^2 \phi + \cos^2 \phi)}
 \end{aligned}$$

$$h_3 = r \sin \theta$$

~~Scale~~ Scale factors for Spherical System is given by

$$\left. \begin{aligned} h_1 &= 1 \\ h_2 &= r \\ h_3 &= r \sin \theta \end{aligned} \right\} .$$

orthogonality of cylindrical system

let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector

In cylindrical system

$\vec{r} = \rho \cos \phi \hat{i} + \rho \sin \phi \hat{j} + z \hat{k}$ be the position vector

let $\hat{e}_\rho$ $\hat{e}_\phi$ $\hat{e}_z$ be the unit vectors in cylindrical system

$$\hat{e}_\rho = \frac{\frac{\partial \vec{r}}{\partial \rho}}{\left| \frac{\partial \vec{r}}{\partial \rho} \right|} = \frac{\frac{\partial \vec{r}}{\partial \rho}}{h_1} = \frac{\cos \phi \hat{i} + \sin \phi \hat{j} + 0 \hat{k}}{1}$$

$$\boxed{\hat{e}_\rho = \cos \phi \hat{i} + \sin \phi \hat{j}}$$

$$\hat{e}_\phi = \frac{\left| \frac{\partial \vec{r}}{\partial \phi} \right|}{\left| \frac{\partial \vec{r}}{\partial \phi} \right|} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{h_2} = \frac{-\rho \sin \phi \hat{i} + \rho \cos \phi \hat{j} + 0 \hat{k}}{\rho}$$

$$\boxed{\hat{e}_\phi = -\sin \phi \hat{i} + \cos \phi \hat{j}}$$

$$\hat{e}_z = \frac{\frac{\partial \vec{r}}{\partial z}}{\left| \frac{\partial \vec{r}}{\partial z} \right|} = \frac{\frac{\partial \vec{r}}{\partial z}}{h_3} = \frac{0 \hat{i} + 0 \hat{j} + \hat{k}}{1}$$

$$\boxed{\hat{e}_z = \hat{k}}$$

consider

$$\begin{aligned} \hat{e}_r \cdot \hat{e}_\phi &= (\cos \phi \hat{i} + \sin \phi \hat{j}) \cdot (-\sin \phi \hat{i} + \cos \phi \hat{j}) \\ &= -\cos \phi \sin \phi + \sin \phi \cos \phi \end{aligned}$$

$$\boxed{\hat{e}_r \cdot \hat{e}_\phi = 0}$$

consider

$$\begin{aligned} \hat{e}_\phi \cdot \hat{e}_z &= (-\sin \phi \hat{i} + \cos \phi \hat{j}) \cdot (\hat{k}) \\ &= 0 + 0 + 0 \end{aligned}$$

$$\boxed{\hat{e}_\phi \cdot \hat{e}_z = 0}$$

consider

$$\begin{aligned} \hat{e}_z \cdot \hat{e}_r &= \hat{k} \cdot (\cos \phi \hat{i} + \sin \phi \hat{j}) \\ &= 0 + 0 + 0 \end{aligned}$$

$$\boxed{\hat{e}_z \cdot \hat{e}_r = 0} \Rightarrow \text{proves orthogonality of cylindrical system}$$

orthogonality of Spherical System

let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector

$\Rightarrow \vec{r} = r \sin\theta \cos\phi \hat{i} + r \sin\theta \sin\phi \hat{j} + r \cos\theta \hat{k}$ be the vector in Spherical System.

let $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ be the unit vectors in Spherical System

At w.k.t

$$\hat{e}_r = \frac{\frac{\partial \vec{r}}{\partial r}}{h_1} = \frac{\sin\theta \cos\phi \hat{i} + \sin\theta \sin\phi \hat{j} + \cos\theta \hat{k}}{1}$$

$$\boxed{\hat{e}_r = \sin\theta \cos\phi \hat{i} + \sin\theta \sin\phi \hat{j} + \cos\theta \hat{k}}$$

$$\hat{e}_\theta = \frac{\frac{\partial \vec{r}}{\partial \theta}}{h_2} = \frac{r \cos\theta \cos\phi \hat{i} + r \cos\theta \sin\phi \hat{j} + r \cancel{\cos\theta} \hat{k}}{r} \quad (\sin\theta)$$

$$\boxed{\hat{e}_\theta = \cos\theta \cos\phi \hat{i} + \cos\theta \sin\phi \hat{j} - \sin\theta \hat{k}}$$

$$\hat{e}_\phi = \frac{\frac{\partial \vec{r}}{\partial \phi}}{h_3} = \frac{-r \sin\theta \sin\phi \hat{i} + r \sin\theta \cos\phi \hat{j} + r \cancel{\cos\theta} \hat{k}}{r \sin\theta}$$

$$\boxed{\hat{e}_\phi = -\sin\phi \hat{i} + \cos\phi \hat{j}}$$

consider

$$\hat{e}_r \cdot \hat{e}_\theta = (\sin\theta \cos\phi \hat{i} + \sin\theta \sin\phi \hat{j} + \cos\theta \hat{k}) \cdot (\cos\theta \cos\phi \hat{i} + \cos\theta \sin\phi \hat{j} - \sin\theta \hat{k})$$

$$\begin{aligned} \hat{e}_r \cdot \hat{e}_\theta &= \sin\theta \cos\theta \cos^2\phi + \sin\theta \cos\theta \sin^2\phi - \sin\theta \cos\theta \\ &= \sin\theta \cos\theta (\cos^2\phi + \sin^2\phi) - \sin\theta \cos\theta \end{aligned}$$

$$\boxed{\hat{e}_r \cdot \hat{e}_\theta = 0}$$

$$\hat{e}_\theta \cdot \hat{e}_\phi = (\cos\theta \cos\phi \hat{i} + \cos\theta \sin\phi \hat{j} - \sin\theta \hat{k}) \cdot (-r \sin\phi \hat{i} + r \cos\phi \hat{j}) \quad (9)$$

$$\hat{e}_\theta \cdot \hat{e}_\phi = -r \cancel{\cos\theta \cos\phi} \sin\phi + r \cancel{\cos\theta \sin\phi} \cos\phi + 0$$

$$\boxed{\hat{e}_\theta \cdot \hat{e}_\phi = 0}$$

Consider

$$\hat{e}_\phi \cdot \hat{e}_r = (-r \sin\phi \hat{i} + r \cos\phi \hat{j}) \cdot (\sin\theta \cos\phi \hat{i} + \sin\theta \sin\phi \hat{j} + \cos\theta \hat{k})$$

$$\hat{e}_\phi \cdot \hat{e}_r = -r \sin\theta \cancel{\sin\phi \cos\phi} + r \sin\theta \cancel{\sin\phi \cos\phi} + 0$$

$$\boxed{\hat{e}_\phi \cdot \hat{e}_r = 0}$$

~~proves~~ prove orthogonality of
Spherical system

problems,

10

Q) Express the vector $\vec{A} = z\hat{i} - 2x\hat{j} + y\hat{k}$ in cylindrical co-ordinates

Sol: let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector

Wkt relation b/w cartesian & cylindrical system is given by

$$x = \rho \cos\phi, \quad y = \rho \sin\phi, \quad z = z$$

$$\Rightarrow \vec{r} = \rho \cos\phi \hat{i} + \rho \sin\phi \hat{j} + z \hat{k}$$

let $\vec{A} = A_1 \hat{e}_\rho + A_2 \hat{e}_\phi + A_3 \hat{e}_z$ be the required vector in cylindrical form
①

Where $A_1 = \vec{A} \cdot \hat{e}_\rho$

$$A_2 = \vec{A} \cdot \hat{e}_\phi$$

$$A_3 = \vec{A} \cdot \hat{e}_z$$

Wkt the unit tangent vectors are given by

$$\hat{e}_\rho = \frac{\partial \vec{r} / \partial \rho}{h_1}$$

$$\hat{e}_\phi = \frac{\partial \vec{r} / \partial \phi}{h_2}$$

$$\hat{e}_z = \frac{\partial \vec{r} / \partial z}{h_3}$$

Where $h_1 = 1, h_2 = \rho, h_3 = 1$ are scale factors in cylindrical system

$$\hat{e}_\rho = \frac{\cos\phi \hat{i} + \sin\phi \hat{j} + 0 \hat{k}}{1} \Rightarrow \hat{e}_\rho = \cos\phi \hat{i} + \sin\phi \hat{j}$$

$$\hat{e}_\phi = \frac{-\rho \sin\phi \hat{i} + \rho \cos\phi \hat{j} + 0 \hat{k}}{\rho} \Rightarrow \hat{e}_\phi = -\sin\phi \hat{i} + \cos\phi \hat{j}$$

$$\hat{e}_z = \frac{0 \hat{i} + 0 \hat{j} + 1 \hat{k}}{1} \Rightarrow \hat{e}_z = \hat{k}$$

Given $\vec{A} = z\hat{i} - 2x\hat{j} + y\hat{k}$

$\Rightarrow \vec{A} = z\hat{i} - 2\rho\cos\phi\hat{j} + \rho\sin\phi\hat{k}$

Now, $A_1 = \vec{A} \cdot \hat{e}_\rho$

$A_1 = (z\hat{i} - 2\rho\cos\phi\hat{j} + \rho\sin\phi\hat{k}) \cdot (\cos\phi\hat{i} + \sin\phi\hat{j})$

$A_1 = z\cos\phi - 2\rho\sin\phi\cos\phi$

Wdy

$A_2 = \vec{A} \cdot \hat{e}_\phi$

$A_2 = (z\hat{i} - 2\rho\cos\phi\hat{j} + \rho\sin\phi\hat{k}) \cdot (-\sin\phi\hat{i} + \cos\phi\hat{j})$

$A_2 = -z\sin\phi - 2\rho\cos^2\phi$

$A_3 = \vec{A} \cdot \hat{e}_z$

$A_3 = (z\hat{i} - 2\rho\cos\phi\hat{j} + \rho\sin\phi\hat{k}) \cdot (\hat{k})$

$A_3 = \rho\sin\phi$

$\therefore \vec{A} \rightarrow$

$\vec{A} = (z\cos\phi - 2\rho\sin\phi\cos\phi)\hat{e}_\rho + (-z\sin\phi - 2\rho\cos^2\phi)\hat{e}_\phi + (\rho\sin\phi)\hat{e}_z$

Q₂ Represent the vector $\vec{A} = 2x\hat{i} - 3y^2\hat{j} + xz\hat{k}$ in cylindrical polar co-ordinate system.

Solⁿ $\vec{A} = (2\rho\cos^2\phi - 3\rho^2\sin^2\phi)\hat{e}_\rho + (-2\rho\sin\phi\cos\phi - 3\rho^2\sin^2\phi\cos\phi)\hat{e}_\phi + (\rho\cos\phi z)\hat{e}_z$

Q. Express $\vec{F} = y\hat{i} - z\hat{j} + x\hat{k}$ in Spherical polar Co-ordinates & hence find F_r, F_θ, F_ϕ

Sol. let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector

Let relation b/w cartesian & spherical system is given by

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$\text{Now } \vec{r} = r \sin \theta \cos \phi \hat{i} + r \sin \theta \sin \phi \hat{j} + r \cos \theta \hat{k}$$

let $\vec{F} = F_1 \hat{e}_r + F_2 \hat{e}_\theta + F_3 \hat{e}_\phi$ be the required vector in Spherical form

$$\text{where } F_1 = \vec{F} \cdot \hat{e}_r$$

$$F_2 = \vec{F} \cdot \hat{e}_\theta$$

$$F_3 = \vec{F} \cdot \hat{e}_\phi$$

and unit tangent vectors are given by

$$\hat{e}_r = \frac{\partial \vec{r} / \partial r}{h_1}$$

$$\hat{e}_\theta = \frac{\partial \vec{r} / \partial \theta}{h_2}$$

$$\hat{e}_\phi = \frac{\partial \vec{r} / \partial \phi}{h_3}$$

here $h_1 = 1$, $h_2 = r$, $h_3 = r \sin \theta$ are scale factors in Spherical system

$$\hat{e}_r = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}$$

$$\boxed{\hat{e}_r = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}}$$

$$\hat{e}_\theta = \frac{r \cos \theta \cos \phi \hat{i} + r \cos \theta \sin \phi \hat{j} + (-r \sin \theta) \hat{k}}{r}$$

$$\left\langle \hat{e}_\theta = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k} \right\rangle$$

$$\hat{e}_\phi = \frac{-r \sin \theta \sin \phi \hat{i} + r \sin \theta \cos \phi \hat{j} + 0 \hat{k}}{r \sin \theta}$$

$$\left\langle \hat{e}_\phi = -\sin \phi \hat{i} + \cos \phi \hat{j} \right\rangle$$

Now we have $\vec{F} = y \hat{i} - z \hat{j} + x \hat{k}$

$$\vec{F}_1 = \vec{F} \cdot \hat{e}_\theta$$

$$\Rightarrow \vec{F} = r \sin \theta \sin \phi \hat{i} - r \cos \theta \hat{j} + r \sin \theta \cos \phi \hat{k}$$

Also we have

$$\vec{F}_1 = F_1 = \vec{F} \cdot \hat{e}_\theta$$

$$F_1 = (r \sin \theta \sin \phi \hat{i} - r \cos \theta \hat{j} + r \sin \theta \cos \phi \hat{k}) \cdot (\sin \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k})$$

$$F_1 = r \sin^2 \theta \sin \phi \cos \phi - r \sin \theta \cos \theta \sin \phi + r \sin \theta \cos \theta \cos \phi.$$

Wdy $F_2 = \vec{F} \cdot \hat{e}_\phi$

$$F_2 = (r \sin \theta \sin \phi \hat{i} - r \cos \theta \hat{j} + r \sin \theta \cos \phi \hat{k}) \cdot (\cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k})$$

$$F_2 = r \sin \theta \cos \theta \sin \phi \cos \phi - r \cos^2 \theta \sin \phi - r \sin^2 \theta \cos \phi.$$

$$F_3 = \vec{F} \cdot \hat{e}_\phi$$

$$F_3 = (r \sin \theta \sin \phi \hat{i} - r \cos \theta \hat{j} + r \sin \theta \cos \phi \hat{k}) \cdot (-\sin \phi \hat{i} + \sin \phi \hat{j})$$

$$F_3 = -r \sin \theta \sin^2 \phi - r \sin \phi \cos \theta$$

$\therefore$

$$\vec{F} = F_1 \hat{e}_r + F_2 \hat{e}_\theta + F_3 \hat{e}_\phi$$

$$\begin{aligned} \vec{F} = & (r \sin^2 \theta \sin \phi \cos \phi - r \sin \theta \cos \theta \sin \phi + r \sin \theta \cos \theta \cos \phi) \hat{e}_r + \\ & (r \sin \theta \cos \theta \sin \phi \cos \phi - r \cos^2 \theta \sin \phi - r \sin^2 \theta \cos \phi) \hat{e}_\theta + \\ & (-r \sin \theta \sin^2 \phi - r \sin \phi \cos \theta) \hat{e}_\phi. \end{aligned}$$

required vector

Module - 3

System of linear equation, Eigen value and Eigen vector

Elementary transformation associated with a matrix

- Interchange of any 2 rows
- Multiplication of any row by a non-zero constant
- Addition to any row a constant multiple of any other row.

Echelon form of a matrix

A non zero matrix A is said to be in row echelon form if the following conditions prevail :-

- All the zero rows are below non zero rows.
- The first non-zero entry in any non zero row is 1.

Ex:-
$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rank of a matrix

The rank of a matrix A in its echelon form is equal to the number of non-zero rows.

It is denoted by $\rho(A)$.

Ex:- If $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}_{2 \times 2}$

$R_2 \rightarrow R_2 - 3R_1$

$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$$\text{If } A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 3 \\ 1 & 5 & 7 \end{bmatrix}_{3 \times 3}$$

$$R_2 \rightarrow 2R_2 + R_1$$

$$R_3 \rightarrow 2R_3 - R_1$$

$$\sim \begin{bmatrix} 2 & 3 & 4 \\ 0 & 7 & 10 \\ 0 & 7 & 10 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 2 & 3 & 4 \\ 0 & 7 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 2$$

Find the rank of the following matrices by applying elementary row transformation

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}_{3 \times 4}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$2/ A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}_{3 \times 4}$$

We shall interchange R_1 either with R_2 or R_3 as the first element in R_1 is zero.

$$R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 2$$

$$3/ A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \end{bmatrix}_{3 \times 4}$$

$$R_2 \rightarrow 2R_2 - R_1$$

$$R_3 \rightarrow 2R_3 - R_1$$

$$\sim \begin{bmatrix} 4 & 0 & 2 & 1 \\ 0 & 2 & 4 & 7 \\ 0 & 6 & 6 & 13 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_2$$

$$\sim \begin{bmatrix} 4 & 0 & 2 & 1 \\ 0 & 2 & 4 & 7 \\ 0 & 0 & -6 & -8 \end{bmatrix}$$

$$\rho(A) = 3$$

4/6 $A = \begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}_{4 \times 4}$

$$R_2 \rightarrow 2R_2 - R_1$$

$$R_3 \rightarrow 2R_3 - R_1$$

$$\sim \begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 5 & 9 & -1 \\ 0 & 1 & 5 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$R_3 \rightarrow 5R_3 - R_2$$

$$R_4 \rightarrow 5R_4 - R_2$$

$$\sim \begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 5 & 9 & -1 \\ 0 & 0 & 16 & 16 \\ 0 & 0 & -4 & -4 \end{bmatrix}$$

$$R_3 \rightarrow R_3/16, R_4 \rightarrow R_4/-4$$

$$\sim \begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 5 & 9 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - R_3$$

$$\sim \begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 5 & 9 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 3$$

$$5) A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}_{4 \times 4}$$

$$R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$R_4 \rightarrow R_4 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$R_4 \rightarrow R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2$$

$$6) A = \begin{bmatrix} 8 & 2 & 1 & 6 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & 3 \\ 5 & 1 & 1 & 4 \end{bmatrix}_{4 \times 4}$$

$$R_2 \rightarrow 4R_2 - R_1$$

$$R_3 \rightarrow 8R_3 - 3R_1$$

$$R_4 \rightarrow 8R_4 - 5R_1$$

$$\sim \begin{bmatrix} 8 & 2 & 1 & 6 \\ 0 & 2 & -1 & -2 \\ 0 & -6 & 5 & 6 \\ 0 & -2 & 3 & 2 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 3R_2$$

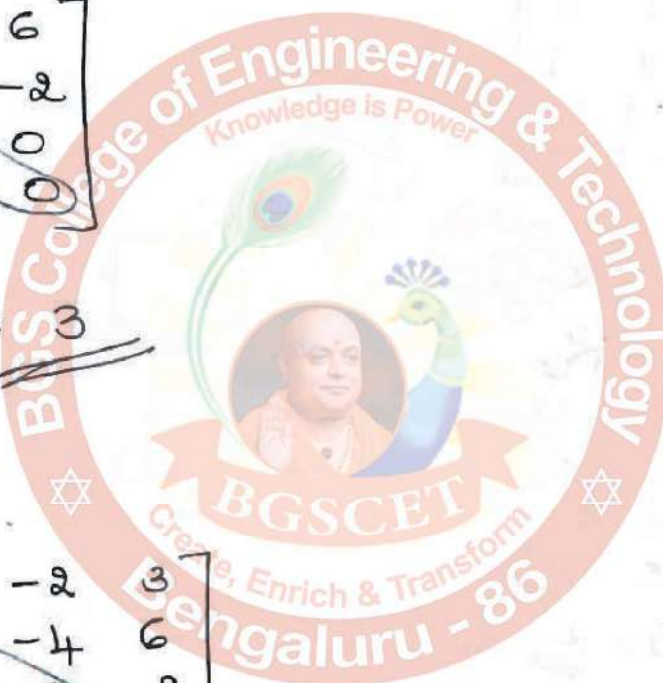
$$R_4 \rightarrow R_4 + R_2$$

$$\sim \begin{bmatrix} 8 & 2 & 1 & 6 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - R_3$$

$$\sim \begin{bmatrix} 8 & 2 & 1 & 6 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 3$$



$$\therefore A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 2 & 5 & -4 & 6 \\ -1 & -3 & 2 & -2 \\ 2 & 4 & -1 & 6 \end{bmatrix} \quad 4 \times 4$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 + R_1$$

$$R_4 \rightarrow R_4 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

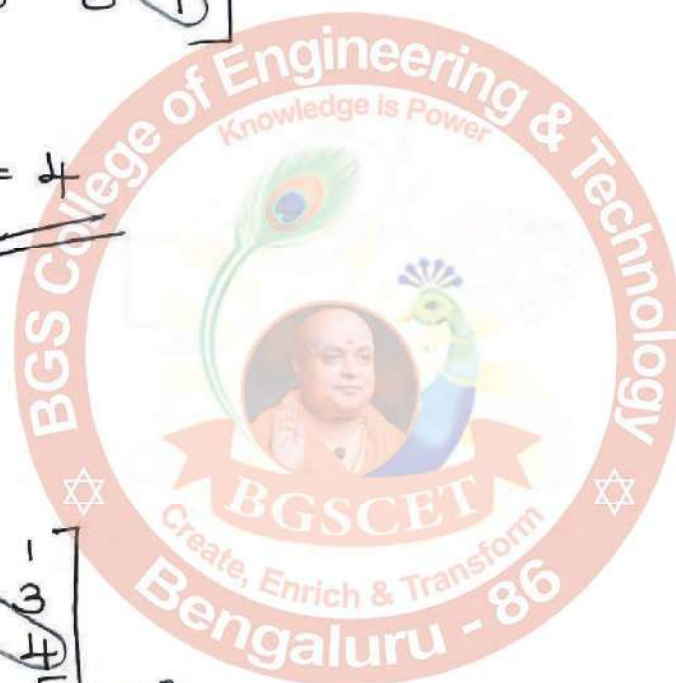
$$R_3 \rightarrow R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$R_4 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rho(A) = 4$$



$$\text{Ex } A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 4 \\ 2 & 1 & 5 \end{bmatrix}_{4 \times 3}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$R_4 \rightarrow R_4 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & -3 & 3 \end{bmatrix}$$

$$R_4 \rightarrow R_4/3$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$R_4 \rightarrow R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 2$$

Consistency of a system of linear Equation

Consider a system of 'm' equations in 'n' unknowns represented in the matrix form $AX = B$.

where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

A is called co-eff matrix.

Now $[A : B] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & : & b_2 \\ \vdots & \vdots & \ddots & \vdots & : & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & : & b_m \end{bmatrix}$

The system of equations is represented by the matrix eqn $AX = B$ is consistent if $\rho(A) = \rho(A : B)$

If $\rho(A) = \rho(A : B) = n$ then

of Unique solⁿ :- $\rho(A) = \rho(A : B) = r = n$ where n is no of unknowns

of Infinite solⁿ :- $\rho(A) = \rho(A : B) = r < n$ then $(n-r)$ unknowns can take arbitrary values.

If $\rho(A) \neq \rho(A : B)$ then the system is inconsistent.

1) Test for consistency and solve

$$x + y + z = 6$$

$$x - y + 2z = 5$$

$$3x + y + z = 8$$

The augmented matrix is

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & -1 & 2 & : & 5 \\ 3 & 1 & 1 & : & 8 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & -2 & -2 & : & -10 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 1 & : & -1 \\ 0 & 0 & -3 & : & -9 \end{bmatrix}$$

$$\rho(A) = 3, \rho(A:B) = 3 \Rightarrow r = 3$$

$$n = 3, \rho(A) = \rho(A:B)$$

$$\therefore \infty = n = 3$$

The given system is consistent and has unique solⁿ.

$$x + y + z = 6, \quad -2y + z = -1, \quad -3z = -9$$

$$x + 2 + 3 = 6$$

$$-2y + 3 = -1$$

$$z = 3$$

$$x = 1$$

$$y = 2$$

$$5x_1 + x_2 + 3x_3 = 20$$

$$2x_1 + 5x_2 + 2x_3 = 18$$

$$3x_1 + 2x_2 + x_3 = 14$$

The augmented matrix is

$$[A:B] = \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 2 & 5 & 2 & : & 18 \\ 3 & 2 & 1 & : & 14 \end{bmatrix}$$

$$R_2 \rightarrow 5R_2 - 2R_1$$

$$R_3 \rightarrow 5R_3 - 3R_1$$

$$\sim \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 0 & 23 & 4 & : & 50 \\ 0 & 7 & -4 & : & 10 \end{bmatrix}$$

$$R_3 \rightarrow 23R_3 - 7R_2$$

$$\sim \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 0 & 23 & 4 & : & 50 \\ 0 & 0 & -120 & : & -120 \end{bmatrix}$$

$$\rho(A) = 3, \quad \rho(A:B) = 3 \Rightarrow r = 3$$

$$n = 3, \quad \rho(A) = \rho(A:B)$$

$$\therefore r = n = 3$$

The given system of eq is consistent and has unique sol.

$$5x_1 + x_2 + 3x_3 = 20, \quad 23x_2 + 4x_3 = 50, \quad -120x_3 = -120$$

$$5x_1 + 2 + 3 = 20$$

$$23x_2 + 4 = 50$$

$$x_3 = 1$$

$$x_2 = 2$$

$$x_1 = 3$$

$$\begin{aligned} x + 2y + 2z &= 1 \\ 2x + y + z &= 2 \\ 3x + 2y + 2z &= 3 \\ y + z &= 0 \end{aligned}$$

The augmented matrix is

$$[A:B] = \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 2 & 1 & 1 & : & 2 \\ 3 & 2 & 2 & : & 3 \\ 0 & 1 & 1 & : & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & -3 & -3 & : & 0 \\ 0 & -4 & -4 & : & 0 \\ 0 & 1 & 1 & : & 0 \end{bmatrix}$$

$$R_2 \rightarrow R_2 \mid -3$$

$$R_3 \rightarrow R_3 \mid -4$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & 1 & 1 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 1 & 1 & : & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$R_4 \rightarrow R_4 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$\rho(A) = 2, \quad \rho(A:B) = 2 \Rightarrow r = 2$$

$$n = 3, \quad \rho(A) = \rho(A:B)$$

$$\therefore r < n \Rightarrow 2 < 3$$

The system is consistent and has infinite solⁿ.

Here $n - r = 3 - 2 = 1$ arbitrary

$$x + 2y + 2z = 1, \quad y + z = 0$$

$$\text{Let } \boxed{z = K}$$

$$4x + 2y + 3z = 14$$

$$4x + 5y + 7z = 35$$

$$3x + 3y + 4z = 21$$

The augmented matrix is

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 4 & 5 & 7 & 35 \\ 3 & 3 & 4 & 21 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 4R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & -3 & -5 & -21 \\ 0 & -3 & -5 & -21 \end{array} \right]$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 14 \\ 0 & -3 & -5 & -21 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\rho(A) = 2, \quad \rho(A:B) = 2 \Rightarrow r = 2$$

$$n = 3, \quad \rho(A) = \rho(A:B)$$

$$\therefore r < n \Rightarrow 2 < 3$$

The system is consistent and has infinite solⁿ.

$$\text{Here } n - r = 3 - 2 = 1 \text{ arbitrary}$$

Hence one of the variables can take arbitrary values.

$$x + 2y + 3z = 14, \quad -3y - 5z = -21$$

$$\text{Let } z = K$$

$$-3y - 5K = -21$$

$$y = \frac{21 - 5K}{3}$$

$$\begin{aligned} x - 4y + 7z &= 14 \\ 3x + 8y - 2z &= 13 \\ 7x - 8y + 26z &= 5 \end{aligned}$$

The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 3 & 8 & -2 & : & 13 \\ 7 & -8 & 26 & : & 5 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1$$

$$R_3 \rightarrow R_3 - 7R_1$$

$$\sim \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 0 & 20 & -23 & : & -29 \\ 0 & 20 & -23 & : & -93 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -4 & 7 & : & 14 \\ 0 & 20 & -23 & : & -29 \\ 0 & 0 & 0 & : & -64 \end{bmatrix}$$

$$\rho(A) = 2, \quad \rho(A : B) = 3$$

Since $\rho(A) \neq \rho(A : B)$ the system is inconsistent

Hence no sol.

Ex: S.T the foll system of equations does not possess any solⁿ

$$5x + 3y + 7z = 5$$

$$3x + 26y + 2z = 9$$

$$7x + 2y + 10z = 5$$

The augmented matrix is

$$[A:B] = \left[\begin{array}{ccc|c} 5 & 3 & 7 & 5 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array} \right]$$

$$R_2 \rightarrow 5R_2 - 3R_1$$

$$R_3 \rightarrow 5R_3 - 7R_1$$

$$\sim \left[\begin{array}{ccc|c} 5 & 3 & 7 & 5 \\ 0 & 121 & -11 & 30 \\ 0 & -11 & 1 & -10 \end{array} \right]$$

$$R_3 \rightarrow 11R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|c} 5 & 3 & 7 & 5 \\ 0 & 121 & -11 & 30 \\ 0 & 0 & 0 & -80 \end{array} \right]$$

$$\rho(A) = 2, \quad \rho(A:B) = 3$$

Since $\rho(A) \neq \rho(A:B)$ the system is inconsistent
Hence no solⁿ.

If Investigate the values of λ & μ such that the system of eqⁿ

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= \mu \end{aligned}$$

may have

of unique solⁿ
 of infinite solⁿ
 of No solⁿ

The augmented matrix is $[A:B]$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 3 & : & 10 \\ 1 & 2 & \lambda & : & \mu \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 1 & \lambda - 1 & : & \mu - 6 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 2 & : & 4 \\ 0 & 0 & \lambda - 3 & : & \mu - 10 \end{bmatrix}$$

if We must have $\rho(A) = \rho(A:B) = 3$. $\rho(A) = 3 \iff \lambda - 3 \neq 0$
 ie $\lambda \neq 3$ irrespective the value of μ , $\rho(A:B)$
 will also be 3.

Thus unique solⁿ when $\lambda \neq 3$.

if Here $n = 3$ & we need $\rho(A) = \rho(A:B) = 2 < 3$
 $\therefore \rho(A) = \rho(A:B) = 2$ only when the last row of
 $[A:B]$ is completely zero. This is possible if
 $\lambda - 3 = 0$ & $\mu - 10 = 0$ ie $\lambda = 3$ & $\mu = 10$.

if We must have $\rho(A) \neq \rho(A:B)$
 $\rho(A) = 3 \iff \lambda \neq 3$ & hence if $\lambda = 3$ we get $\rho(A) = 2$

Q Find for what values of K the system of equations

$$x + y + z = 1$$

$$x + 2y + 4z = K$$

$$x + 4y + 10z = K^2$$

possesses a solⁿ. Solve completely in each case.

The augmented matrix is $[A:B]$

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 1 & 2 & 4 & : & K \\ 1 & 4 & 10 & : & K^2 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & K-1 \\ 0 & 3 & 9 & : & K^2-1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 1 \\ 0 & 1 & 3 & : & K-1 \\ 0 & 0 & 0 & : & K^2-3K+2 \end{bmatrix}$$

$\rho(A) = 2$ & system is consistent if $\rho(A:B)$ is also 2

This is possible if $K^2 - 3K + 2 = 0 \therefore K = 1, K = 2$

Hence we conclude that the system possesses a solⁿ if $K = 1, 2$

Since $\rho(A) = \rho(A:B) = 2 < 3$ for the cases $K = 1, 2$ the system will have infinite solⁿ

Case 1 :- $K = 1$ then $x + y + z = 1$
 $y + 3z = 0$

Let $z = K_1$ then $y = -3K_1$ & $x = 1 + 2K_1$

Case 2 :- $K = 2$ then $x + y + z = 1$

Gauss - Elimination Method

Solve the foll system of equations by Gauss Elimination Method

$$1) \quad x + y + z = 9$$

$$x - 2y + 3z = 8$$

$$2x + y - z = 3$$

The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 1 & -2 & 3 & : & 8 \\ 2 & 1 & -1 & : & 3 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -3 & 2 & : & -1 \\ 0 & -1 & -3 & : & -15 \end{bmatrix}$$

$$R_3 \rightarrow 3R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -3 & 2 & : & -1 \\ 0 & 0 & -11 & : & -44 \end{bmatrix}$$

Hence, $-11z = -44$

$$(z = 4)$$

$$-3y + 2z = -1$$

$$-3y + 8 = -1$$

$$(y = 3)$$

$$x + y + z = 9$$

$$x + 3 + 4 = 9$$

$$(x = 2)$$

$$\begin{aligned} 2x + y + 4z &= 12 \\ 4x + 11y - z &= 33 \\ 8x - 3y + 2z &= 20 \end{aligned}$$

The augmented matrix is

$$[A:B] = \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 4 & 11 & -1 & 33 \\ 8 & -3 & 2 & 20 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 4R_1$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 9 & -9 & 9 \\ 0 & -7 & -14 & -28 \end{array} \right]$$

$$R_2 \rightarrow R_2/9, \quad R_3 \rightarrow R_3/-7$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 2 & 4 \end{array} \right]$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & 3 \end{array} \right]$$

$$3z = 3$$

$$z = 1$$

$$y - z = 1$$

$$y - 1 = 1$$

$$y = 2$$

$$2x + y + 4z = 12$$

$$2x + 2 + 4 = 12$$

$$x = 3$$

$$3) \begin{cases} x_1 - 2x_2 + 3x_3 = 2 \\ 3x_1 - x_2 + 4x_3 = 4 \\ 2x_1 + x_2 - 2x_3 = 5 \end{cases}$$

$$3x_1 - x_2 + 4x_3 = 4$$

$$2x_1 + x_2 - 2x_3 = 5$$

The augmented matrix is

$$[A:B] = \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 3 & -1 & 4 & 4 \\ 2 & 1 & -2 & 5 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 3R_1$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 5 & -5 & -2 \\ 0 & 5 & -8 & 1 \end{array} \right]$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 5 & -5 & -2 \\ 0 & 0 & -3 & 3 \end{array} \right]$$

Hence $-3x_3 = 3$, $5x_2 - 5x_3 = -2$, $x_1 - 2x_2 + 3x_3 = 2$
 $x_3 = -1$, $5x_2 + 5 = -2$, $x_1 - 2(-1 \cdot 4) + 3(-1) = 2$
 $x_2 = -1.4$, $x_1 = 2.2$

$$4) \begin{cases} 2x + 5y + 7z = 52 \\ 2x + y - z = 0 \\ x + y + z = 9 \end{cases}$$

$$\left. \begin{array}{l} x=1 \\ y=3 \\ z=5 \end{array} \right\}$$

Gauss - Jordan Method

Solve the foll system of equations by Gauss Jordan Method.

$$\begin{aligned}x + y + z &= 9 \\x - 2y + 3z &= 8 \\2x + y - z &= 3\end{aligned}$$

The augmented matrix is

$$[A : B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & -2 & 3 & 8 \\ 2 & 1 & -1 & 3 \end{array} \right]$$

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & -1 & -3 & -15 \end{array} \right]$$

$$R_3 \rightarrow 3R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & -11 & -44 \end{array} \right]$$

$$R_3 \rightarrow R_3 / -11$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & 1 & 4 \end{array} \right]$$

$$R_1 \rightarrow 3R_1 + R_2$$

$$\sim \left[\begin{array}{ccc|c} 3 & 0 & 5 & 26 \\ 0 & -3 & 2 & -1 \\ 0 & 0 & 1 & 4 \end{array} \right]$$

$$R_1 \rightarrow R_1 - 5R_3$$

$$R_2 \rightarrow R_2 - 2R_3$$

$$\sim \left[\begin{array}{ccc|c} 3 & 0 & 0 & 6 \\ 0 & -3 & 0 & -9 \\ 0 & 0 & 1 & 4 \end{array} \right]$$

$$3x = 6$$

$$x = 2$$

$$-3y = -9$$

$$y = 3$$

$$z = 4$$

$$\begin{aligned} 2x + 5y + 7z &= 52 \\ 2x + y - z &= 0 \\ x + y + z &= 9 \end{aligned}$$

The augmented matrix is

$$[A:B] = \left[\begin{array}{ccc|c} 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \\ 1 & 1 & 1 & 9 \end{array} \right]$$

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow 2R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 2 & 5 & 7 & 52 \\ 0 & -4 & -8 & -52 \\ 0 & -3 & -5 & -34 \end{array} \right]$$

$$R_2 \rightarrow R_2 / -4$$

$$\sim \left[\begin{array}{ccc|c} 2 & 5 & 7 & 52 \\ 0 & 1 & 2 & 13 \\ 0 & -3 & -5 & -34 \end{array} \right]$$

$$R_3 \rightarrow R_3 + 3R_2$$

$$\sim \left[\begin{array}{ccc|c} 2 & 5 & 7 & 52 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

$$R_1 \rightarrow R_1 - 5R_2$$

$$\sim \left[\begin{array}{ccc|c} 2 & 0 & -3 & -13 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

$$R_1 \rightarrow R_1 + 3R_3$$

$$R_2 \rightarrow R_2 - 2R_3$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array} \right]$$

$$2x = 2$$

$$x = 1$$

$$y = 3$$

$$z = 5$$

$$3) \begin{cases} 2x_1 + x_2 + 3x_3 = 1 \\ 4x_1 + 4x_2 + 7x_3 = 1 \\ 2x_1 + 5x_2 + 9x_3 = 3 \end{cases}$$

$$4x_1 + 4x_2 + 7x_3 = 1$$

$$2x_1 + 5x_2 + 9x_3 = 3$$

The augmented matrix is

$$[A:B] = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 4 & 4 & 7 & 1 \\ 2 & 5 & 9 & 3 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 4 & 6 & 2 \end{array} \right]$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 4 & 4 \end{array} \right]$$

$$R_1 \rightarrow 2R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 4 & 0 & 5 & 3 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 4 & 4 \end{array} \right]$$

$$R_1 \rightarrow 4R_1 - 5R_3$$

$$R_2 \rightarrow 4R_2 - R_3$$

$$\sim \left[\begin{array}{ccc|c} 16 & 0 & 0 & -8 \\ 0 & 8 & 0 & -8 \\ 0 & 0 & 4 & 4 \end{array} \right]$$

$$4x_3 = 4, \quad 8x_2 = -8$$

$$x_3 = 1$$

$$x_2 = -1$$

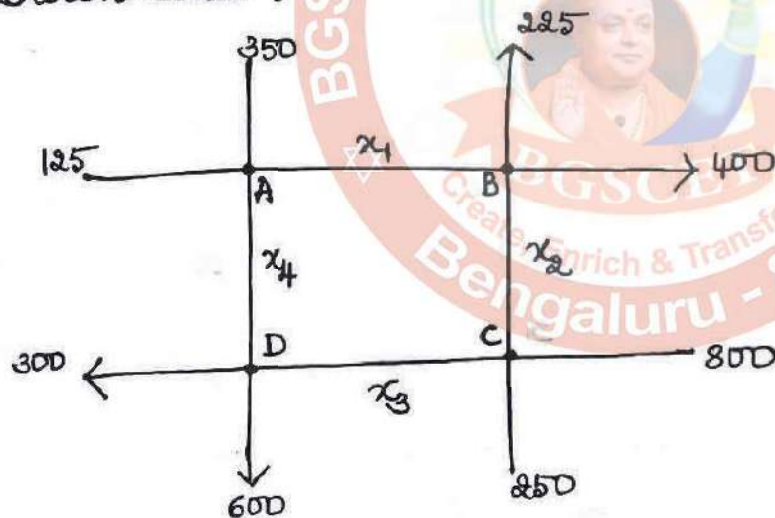
$$16x_1 = -8$$

$$x_1 = -0.5$$

Traffic flow Problems

1. Traffic network can be described by directed networks
2. Intersections corresponds to the nodes of the network.
3. The roads correspond to the directed edges or branch of network. Traffic flows along the branches.
4. Assume all traffic entering an intersection must leave that intersection. "At each node the flow in is equal to the flow out."
5. Assume that the total traffic entering the network equal to the total traffic leaving the network.

1) The flow of traffic through a network of streets is shown below :-



- a) find x_1, x_2, x_3, x_4 to balance the traffic flow
- b) find the traffic flow at $x_4 = 0$
- c) find the traffic flow at $x_4 = 100$

Nodes	flow in	flow out	Equations
A	125+350	$x_1 + x_4$	$x_1 + x_4 = 475$
B	$x_1 + x_2$	400+225	$x_1 + x_2 = 625$
C	800+250	$x_2 + x_3$	$x_2 + x_3 = 1050$
D	$x_3 + x_4$	300+600	$x_3 + x_4 = 900$

The augmented matrix for the linear equations is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 475 \\ 1 & 1 & 0 & 0 & 625 \\ 0 & 1 & 1 & 0 & 1050 \\ 0 & 0 & 1 & 1 & 900 \end{array} \right]$$

$$R_2 \rightarrow R_2 - R_1$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 475 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 1 & 1 & 0 & 1050 \\ 0 & 0 & 1 & 1 & 900 \end{array} \right]$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 475 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 0 & 1 & 1 & 900 \\ 0 & 0 & 1 & 1 & 900 \end{array} \right]$$

$$R_4 \rightarrow R_4 - R_3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 475 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 0 & 1 & 1 & 900 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$x_3 + x_4 = 900, \quad x_2 - x_4 = 150, \quad x_1 + x_4 = 475$$

$$x_3 = 900 - x_4 \quad \hookrightarrow \textcircled{3}, \quad x_2 = 150 + x_4 \quad \hookrightarrow \textcircled{2}, \quad x_1 = 475 - x_4 \quad \hookrightarrow \textcircled{1}$$

$\Rightarrow x_4$ is a free variable

from $\textcircled{1}$

$$x_1 = 475 - x_4$$

from $\textcircled{2}$

$$x_2 = 150 + x_4$$

from $\textcircled{3}$

$$x_3 = 900 - x_4$$

Since all flows must be non negative,
 $\therefore 0 \leq x_4 \leq 475$

b) when $x_4 = 0$

from ①, ②, ③

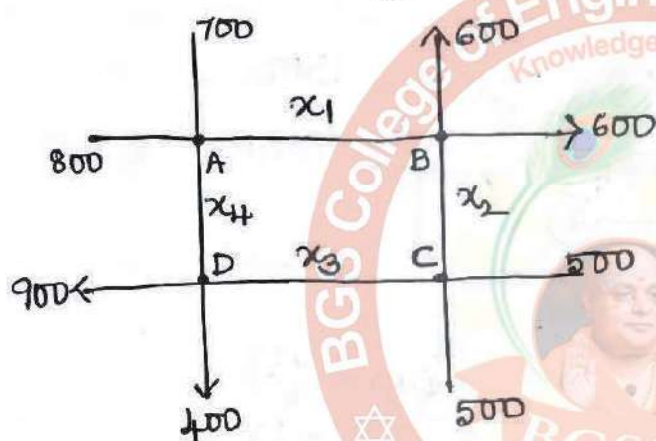
$$x_1 = 475, \quad x_2 = 150, \quad x_3 = 900$$

c) when $x_4 = 100$

from ①, ②, ③

$$x_1 = 375, \quad x_2 = 250, \quad x_3 = 800$$

2) Balance the traffic flow at each intersection



Nodes	Flow in	Flow out	Equations
A	$700 + 800$	$x_1 + x_4$	$x_1 + x_4 = 1500$
B	$x_1 + x_2$	$600 + 600$	$x_1 + x_2 = 1200$
C	$500 + 500$	$x_2 + x_3$	$x_2 + x_3 = 1000$
D	$x_3 + x_4$	$400 + 900$	$x_3 + x_4 = 1300$

The augmented matrix for the linear equations is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1500 \\ 1 & 1 & 0 & 0 & 1200 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 1300 \end{array} \right]$$

$$R_2 \rightarrow R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & : & 1500 \\ 0 & 1 & 0 & -1 & : & -300 \\ 0 & 1 & 1 & 0 & : & 1000 \\ 0 & 0 & 1 & 1 & : & 1300 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & : & 1500 \\ 0 & 1 & 0 & -1 & : & -300 \\ 0 & 0 & 1 & 1 & : & 1300 \\ 0 & 0 & 1 & 1 & : & 1300 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & : & 1500 \\ 0 & 1 & 0 & -1 & : & -300 \\ 0 & 0 & 1 & 1 & : & 1300 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$x_1 + x_4 = 1500, \quad x_2 - x_4 = -300, \quad x_3 + x_4 = 1300$$

$$x_1 = 1500 - x_4, \quad x_2 = -300 + x_4, \quad x_3 = 1300 - x_4$$

①

②

$\Rightarrow x_4$ is a free variable

from ①

$$x_4 \leq 1500$$

from ②

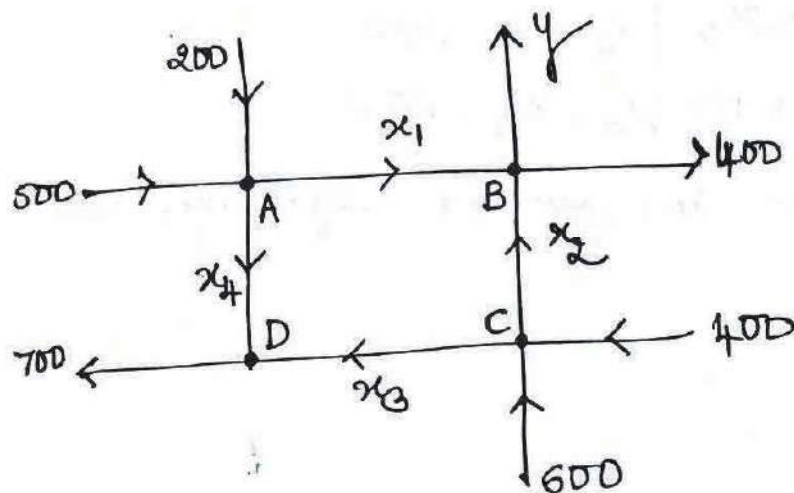
$$x_4 \geq 300$$

from ③

$$x_3 \leq 1300$$

Since all flows must be non-negative
 $\therefore 300 \leq x_4 \leq 1300$

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a) What flow to the north should the traffic light at the intersection B let through to balance the traffic flow in this network?

b) Assuming that the traffic light at B has been set to balance the total flow in and flow out what should the flow be through the inner branches?

Total flow in : $500 + 200 + 400 + 600$

Total flow out : $y + 400 + 700$

To balance the total flow, we have

$$500 + 200 + 400 + 600 = y + 400 + 700$$

$$y = 600$$

Now let's balance the network at each intersection assuming $y = 600$

Node	Flow in	Flow out	Equations
A	$200 + 500$	$x_1 + x_4$	$x_1 + x_4 = 700$
B	$x_1 + x_2$	$600 + 400$	$x_1 + x_2 = 1000$
C	$400 + 600$	$x_2 + x_3$	$x_2 + x_3 = 1000$
D	$x_3 + x_4$	700	$x_3 + x_4 = 700$

The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 700 \\ 1 & 1 & 0 & 0 & 1000 \\ 0 & 1 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 1 & 700 \end{array} \right]$$

$$R_2 \rightarrow R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & : & 700 \\ 0 & 1 & 0 & -1 & : & 300 \\ 0 & 1 & 1 & 0 & : & 1000 \\ 0 & 0 & 1 & 1 & : & 700 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & : & 700 \\ 0 & 1 & 0 & -1 & : & 300 \\ 0 & 0 & 1 & 1 & : & 700 \\ 0 & 0 & 1 & 1 & : & 700 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & : & 700 \\ 0 & 1 & 0 & -1 & : & 300 \\ 0 & 0 & 1 & 1 & : & 700 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Rewriting into linear equations

$$x_1 + x_4 = 700$$

$$x_2 - x_4 = 300$$

$$x_3 + x_4 = 700$$

$$x_1 = 700 - x_4 \quad \text{--- (1)}, \quad x_2 = 300 + x_4 \quad \text{--- (2)}, \quad x_3 = 700 - x_4 \quad \text{--- (3)}$$

$\Rightarrow x_4$ is a free variable

from (1)

from (2)

from (3)

$$x_4 \leq 700, \quad x_4 \geq 0, \quad x_4 \leq 700$$

Since all flows must be non negative

$$0 \leq x_4 \leq 700$$

Eigen values and Eigen Vectors of a square matrix

Given a square matrix A , if there exists a scalar λ (real or complex) and a non zero column matrix X such that $AX = \lambda X$ then λ is called an eigen value of A and X is called an eigen vector of A corresponding to an eigen value λ .

If find the eigen value and the corresponding eigen vector for the following matrices:-

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 4 = 0$$

$$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\lambda^2 - 6\lambda - \lambda + 6 = 0$$

$$\lambda(\lambda - 6) - 1(\lambda - 6) = 0$$

$$(\lambda - 1)(\lambda - 6) = 0$$

$\lambda = 1, 6$ are the eigen values of A

We now form system of equations $\rightarrow$

$$\begin{cases} (5-\lambda)x + 4y = 0 \\ x + (2-\lambda)y = 0 \end{cases} \quad \text{①}$$

$$[A - \lambda I] X = 0$$

$$\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Case 1 :- Put $\lambda = 1$ in ①

$$4x + 4y = 0$$

$$x + y = 0$$

Since both eq^s are same
 $x = -y$

$$\frac{x}{-1} = \frac{y}{1}$$

$$\therefore X_1 = [-1, 1]^T$$

$$\text{② } X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$\therefore X_1$ & X_2 are the eigen vector corresponding to eigen value $\lambda = 1$ & $\lambda = 6$

Case 2 :- Put $\lambda = 6$ in ①

$$-x + 4y = 0$$

$$x - 4y = 0$$

Since both eq^s are same

$$x - 4y = 0$$

$$x = 4y$$

$$\frac{x}{4} = \frac{y}{1}$$

$$\therefore X_2 = [4, 1]^T$$

$$\text{③ } X_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

If $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

The ch. eqⁿ of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 2 \\ -1 & 0-\lambda \end{vmatrix} = 0$$

$$3-\lambda(-\lambda) + 2 = 0$$

$$-3\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda^2 - 2\lambda - \lambda + 2 = 0$$

$$\lambda(\lambda-2) - 1(\lambda-2) = 0$$

We now form system of equation,

$$\begin{cases} (3-\lambda)x + 2y = 0 \\ -x - \lambda y = 0 \end{cases} \quad (1)$$

Case 1 :- put $\lambda = 1$ in (1)

$$\begin{aligned} 2x + 2y &= 0 \\ -x - y &= 0 \end{aligned}$$

Since both eq are same

$$2x + 2y = 0$$

$$2x = -2y$$

$$\frac{x}{-1} = \frac{y}{1}$$

$$\therefore X_1 = [-1, 1]^T$$

$$\textcircled{a} X_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

X_1 & X_2 are the eigen vector corresponding to eigen values $\lambda = 1$ & $\lambda = 2$

$$\text{or } A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

The characteristic eq is $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$(8-\lambda) [(7-\lambda)(3-\lambda) - 16] + 6 [-6(3-\lambda) + 8] + 2 [24 - 2(7-\lambda)] = 0$$

$$\lambda^3 - 18\lambda^2 + 45\lambda = 0$$

$$\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\lambda = 0$$

$$\lambda^2 - 18\lambda + 45 = 0$$

$$\lambda^2 - 15\lambda - 3\lambda + 45 = 0$$

$$\therefore \lambda = 0, 3, 15$$

Case 2 :- put $\lambda = 2$ in (1)

$$x + 2y = 0$$

$$-x - 2y = 0$$

Since both eq are same

$$x + 2y = 0$$

$$x = -2y$$

$$\frac{x}{-2} = \frac{y}{1}$$

$$\therefore X_2 = [-2, 1]^T$$

$$\textcircled{a} X_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

We now form the system of equations

$$(8-\lambda)x - 6y + 2z = 0$$

$$-6x + (7-\lambda)y - 4z = 0$$

$$2x - 4y + (3-\lambda)z = 0$$

} — ①

Case 1 :- let $\lambda = 0$

$$8x - 6y + 2z = 0$$

$$-6x + 7y - 4z = 0$$

$$2x - 4y + 3z = 0$$

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x}{10} = \frac{y}{20} = \frac{z}{20} \Rightarrow \frac{x}{1} = \frac{y}{2} = \frac{z}{2}$$

$$x_1 = [1, 2, 2]^T$$

or

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Case 2 :- let $\lambda = 3$

$$5x - 6y + 2z = 0$$

$$-6x + 4y - 4z = 0$$

$$2x - 4y + 0z = 0$$

$$\frac{x}{\begin{vmatrix} -6 & 4 \\ 2 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x}{16} = \frac{-y}{-8} = \frac{z}{-16} \Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

$$x_2 = [2, 1, -2]^T$$

or

$$x_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Case 3 :- Let $\lambda = 15$

$$-7x - 6y + 2z = 0$$

$$-6x - 8y - 4z = 0$$

$$2x - 4y - 12z = 0$$

$$\frac{x}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$$

$$\frac{x}{40} = \frac{-y}{40} = \frac{z}{20}$$

$$\frac{x}{2} = \frac{-y}{2} = \frac{z}{1}$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$$

$$x_3 = [2, -2, 1]^T$$

$$x_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore x_1, x_2, x_3$ are the eigen vector corresponding to eigen value $\lambda = 0, 3, 15$.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

The characteristic eqⁿ is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ -1 & 4-\lambda & -1 \\ -1 & 2 & 0-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(4-\lambda)(-\lambda) + 2] + 0 + 1[-2 + (4-\lambda)] = 0$$

$$(2-\lambda)[-4\lambda + \lambda^2 + 2] + [-2 + 4 - \lambda] = 0$$

$$(2-\lambda)[\lambda^2 - 4\lambda + 2] + (2 - \lambda) = 0$$

$$3\lambda + \lambda^2 + 2 - \lambda = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda + 6 = 0$$

$\lambda = 1$ is a root by insp,

$$\lambda = 1 \quad \left| \begin{array}{ccc|c} 1 & -6 & 11 & 6 \\ 0 & 1 & -5 & 6 \\ \hline 1 & -5 & 6 & 0 \end{array} \right|$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\lambda^2 - 3\lambda - 2\lambda + 6 = 0$$

$$\lambda(\lambda - 3) - 2(\lambda - 3) = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 2, \lambda = 3$$

$\therefore \lambda = 1, 2, 3$ are the eigen values of A.

We now form the system of equations

$$(2 - \lambda)x + 0y + z = 0$$

$$-x + (4 - \lambda)y - z = 0$$

$$-x + 2y - \lambda z = 0$$

Case 1 :- let $\lambda = 1$

$$x + 0y + z = 0$$

$$-x + 3y - z = 0$$

$$-x + 2y - z = 0$$

$$\frac{x}{-1} = \frac{-y}{0} = \frac{z}{1}$$

$$\left| \begin{array}{cc} 3 & -1 \\ 2 & -1 \end{array} \right| \quad \left| \begin{array}{cc} -1 & -1 \\ -1 & -1 \end{array} \right| \quad \left| \begin{array}{c} -1 \\ -1 \\ 2 \end{array} \right|$$

$$\frac{x}{-1} = \frac{-y}{0} = \frac{z}{1} \Rightarrow \frac{x}{-1} = \frac{y}{0} = \frac{z}{1}$$

$$X_1 = [-1, 0, 1]^T$$

$$\textcircled{2} X_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Case 2 :- Let $\lambda = 2$

$$0x + 0y + z = 0$$

$$-x + 2y - z = 0$$

$$-x + 2y - 2z = 0$$

$$\frac{x}{\begin{vmatrix} 2 & -1 \\ 2 & -2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -1 & -1 \\ -1 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 2 \\ -1 & 2 \end{vmatrix}}$$

$$\frac{x}{-2} = \frac{-y}{1} = \frac{z}{0} \Rightarrow \frac{x}{-2} = \frac{y}{-1} = \frac{z}{0}$$

$$X_2 = [-2, -1, 0]^T \quad \text{or} \quad X_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

Case 3 :- Let $\lambda = 3$

$$-x + 0y + z = 0$$

$$-x + y - z = 0$$

$$-x + 2y - 3z = 0$$

$$\frac{x}{\begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -1 & -1 \\ -1 & -3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix}}$$

$$\frac{x}{-1} = \frac{-y}{2} = \frac{z}{-1} \Rightarrow \frac{x}{-1} = \frac{y}{-2} = \frac{z}{-1}$$

$$X_3 = [-1, -2, -1]^T \quad \text{or} \quad X_3 = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$$

X_1, X_2, X_3 are the eigen vectors corresponding to
eigen values
 $\lambda = 1, 2, 3$

$$\textcircled{1} A = \begin{bmatrix} 11 & -8 & 4 \\ -8 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}$$

$\lambda = -5, -5, 16$ are eigen values

$$x_1 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$$\begin{cases} (11-\lambda)x - 8y + 4z = 0 \\ -8x + (-1-\lambda)y - 2z = 0 \\ 4x - 2y + (-4-\lambda)z = 0 \end{cases} \textcircled{1}$$

Put $\lambda = -5$

$$\begin{cases} 16x - 8y + 4z = 0 \\ -8x + 4y - 2z = 0 \\ 4x - 2y + z = 0 \end{cases}$$

All on one eqⁿ

$$16x - 8y + 4z = 0$$

$$4x - 2y + z = 0$$

$$\begin{aligned} \text{Put } x=0 & \quad \left. \begin{aligned} y=1 \\ 4(0) - 2(1) + z = 0 \\ z = 2 \end{aligned} \right\} \text{Put } x=1 \\ & \quad \left. \begin{aligned} y=0 \\ 4(1) - 2(0) + z = 0 \\ z = -4 \end{aligned} \right\} \end{aligned}$$

$$x_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$$

Put $\lambda = 16$

$$-5x - 8y + 4z = 0$$

$$-8x - 17y - 2z = 0$$

$$4x - 2y - 20z = 0$$

$$\frac{x}{\begin{vmatrix} -8 & 4 \\ -17 & -2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -5 & 4 \\ -8 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -5 & -8 \\ -8 & -17 \end{vmatrix}}$$

$$\frac{x}{84} = \frac{-y}{42} = \frac{z}{21}$$

$$\frac{x}{4} = \frac{-y}{-2} = \frac{z}{1}$$

$$x_1 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

$$\textcircled{2} A = \begin{bmatrix} 5 & 3 \\ 2 & 10 \end{bmatrix}$$

$\lambda = 4, 11$ are eigen values

$$x_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\textcircled{3} A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$\lambda = -2, 6, 3$ are eigen values

$$x_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

5) find the eigen values & eigen vectors of

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

since the matrix is upper triangular matrix
the eigen values are $\lambda = 3, 1, 0$

⑥ The characteristic eqⁿ of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & 4 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 0-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)[(1-\lambda)(-\lambda)] - 4[0] + 2[0] = 0$$

$$(3-\lambda)(-\lambda + \lambda^2) = 0$$

$$-3\lambda + 3\lambda^2 + \lambda^2 - \lambda^3 = 0$$

$$\lambda^3 - 4\lambda^2 + 3\lambda = 0$$

$\lambda = 1$ By insp

$$\lambda = 1 \quad \begin{array}{ccc|c} 1 & -4 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ \hline 1 & -3 & 0 & 0 \end{array}$$

$$\lambda^2 - 3\lambda = 0$$

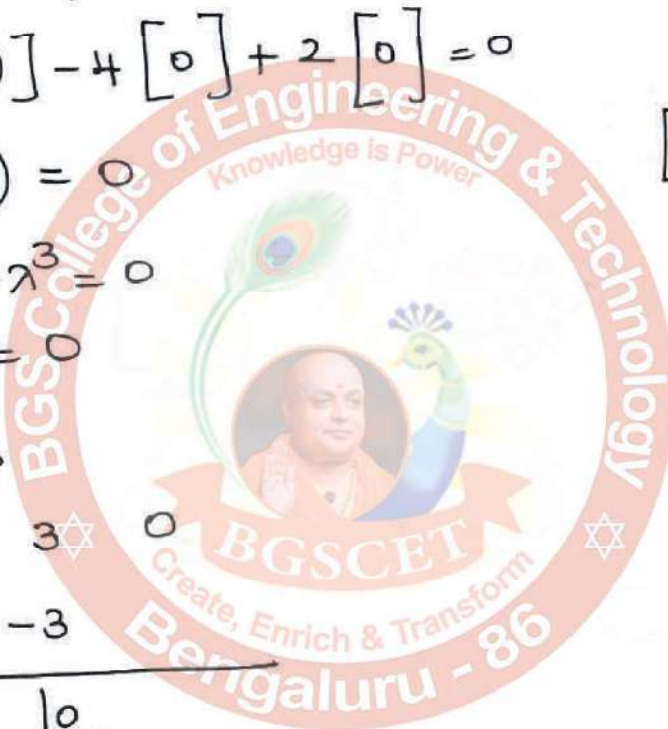
$$\lambda(\lambda - 3) = 0$$

$$\lambda = 0, \lambda = 3$$

$\therefore \lambda = 0, 1, 3$ are the eigen values.

We now form system of equations

[$\because$ since the matrix is singular we can't use Cramer's rule]



$$\left. \begin{aligned} (3-\lambda)x + 4y + 2z &= 0 \\ (1-\lambda)y + 2z &= 0 \end{aligned} \right\} \textcircled{1}$$

Case 1:- Put $\lambda=0$ in $\textcircled{1}$

$$\begin{aligned} 3x + 4y + 2z &= 0 \\ 0x + y + 2z &= 0 \end{aligned}$$

$$\frac{x}{\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{x}{6} = \frac{-y}{6} = \frac{z}{3}$$

$$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$$

$$\therefore X_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Case 2:- Put $\lambda=1$ in $\textcircled{1}$

$$\begin{aligned} 2x + 4y + 2z &= 0 \\ 2z &= 0 \\ \therefore z &= 0 \end{aligned}$$

$$\begin{aligned} 2x + 4y &= 0 \\ 2x &= -4y \\ x &= -2y \end{aligned}$$

$$\begin{aligned} \text{Let } y &= k \\ \therefore x &= -2k \end{aligned}$$

$$\therefore X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Case 3:- Put $\lambda=3$ in $\textcircled{1}$

$$\begin{aligned} 4y + 2z &= 0 \\ -2y + 2z &= 0 \\ \therefore 2z &= 2y \\ \textcircled{a} \quad z &= y \\ 4y + 2y &= 0 \\ 6y &= 0 \\ y &= 0 \\ \therefore z &= 0 \end{aligned}$$

$$\text{Let } x = k$$

$$\therefore X_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$6. A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$$

The ch. eq is $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ 7 & -5-\lambda & 1 \\ 6 & -6 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda) [(-5-\lambda)(2-\lambda) + 6] - 1 [7(2-\lambda) - 6] + 1 [-42 - 6(-5-\lambda)] = 0$$

$$\lambda^3 - 12\lambda + 16 = 0$$

$\lambda = 2$ by insp

$$\lambda = 2 \quad \begin{array}{cccc} 1 & 0 & -12 & 16 \\ 0 & 2 & 4 & -16 \\ \hline 1 & 2 & -8 & 0 \end{array}$$

$$\lambda^2 + 2\lambda - 8 = 0$$

$$\lambda^2 + 4\lambda - 2\lambda - 8 = 0$$

$$\lambda(\lambda + 4) - 2(\lambda + 4) = 0$$

$$\lambda = 2, \lambda = -4$$

$\therefore \lambda = 2, 2, -4$ are eigen values.

We now write system of eqs,

$$\left. \begin{array}{l} (3-\lambda)x - y + z = 0 \\ 7x + (-5-\lambda)y + z = 0 \\ 6x + (-6y) + (2-\lambda)z = 0 \end{array} \right\} \textcircled{1}$$

Case 1 :- Put $\lambda = 2$ in $\textcircled{1}$

$$x - y + z = 0$$

$$7x - 7y + z = 0$$

$$6x - 6y = 0$$

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ -7 & 1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 1 \\ 7 & 1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ 7 & -7 \end{vmatrix}}$$

$$\frac{x}{-6} = \frac{-y}{-6} = \frac{z}{-6}$$

$$\text{Let } x = 0$$

$$-y + z = 0$$

$$-y = -z$$

$$y = z$$

$$\frac{y}{1} = \frac{z}{1}$$

$$\therefore x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(a)

$$\text{Let } y = 0$$

$$x + z = 0$$

$$x = -z$$

$$\frac{x}{-1} = \frac{z}{1}$$

$$\therefore x_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$



Case 3:- $\lambda = -4$ in ①

$$7x - y + z = 0$$

$$7x - y + z = 0 \checkmark$$

$$6x - 6y + 6z = 0 \checkmark$$

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ -6 & 6 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 7 & 1 \\ 6 & 6 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 7 & -1 \\ 6 & -6 \end{vmatrix}}$$

$$\frac{x}{0} = \frac{-y}{36} = \frac{z}{-36}$$

$$\frac{x}{0} = \frac{y}{-6} = \frac{z}{-6}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \end{bmatrix}$$

Similarity of matrices & Diagonalisation of a square matrix

Two square matrices A & B of the same order are said to be similar if there exists a non singular matrix P such that $B = P^{-1}AP$.

If A is a square matrix of order n having n linearly independent eigen vectors then there exists an n^{th} order square matrix P such that $P^{-1}AP$ is a diagonal matrix. P is called the modal matrix.

If Reduce the matrix $A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$ to the diagonal form and hence find A^4 .

The ch. eqⁿ of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} (-1-\lambda) & 3 \\ -2 & (4-\lambda) \end{vmatrix} = 0$$

$$(-1-\lambda)(4-\lambda) + 6 = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$\therefore \lambda = 1, 2$ are the eigen values of A

Consider $[A - \lambda I] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} (-1-\lambda) & 3 \\ -2 & (4-\lambda) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(-1-\lambda)x + 3y = 0$$

$$-2x + (4-\lambda)y = 0$$

Case 1 :- let $\lambda = 1$

$$-2x + 3y = 0$$

$$2x = 3y$$

$$\frac{x}{3} = \frac{y}{2}$$

$\therefore x_1 = (3, 2)^T$ is the eigen vector for $\lambda = 1$

Case 2 :- let $\lambda = 2$

$$-3x + 3y = 0$$

$$x = y$$

$$\frac{x}{1} = \frac{y}{1}$$

$\therefore x_2 = (1, 1)^T$ is the eigen vector for $\lambda = 2$

Modal matrix

$$P = [x_1 \ x_2] = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$|P| = 1 \quad (3-2) = 1$$

$$P^{-1} = \frac{1}{|P|} (\text{adj } P)$$

$$= \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$\text{Now } P^{-1}AP = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \text{Diag}(1, 2)$$

$$\text{Q. If } A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$$

The ch. eq of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} (-19-\lambda) & 7 \\ -42 & (16-\lambda) \end{vmatrix} = 0$$

$$\lambda^2 + 3\lambda - 10 = 0$$

$$(\lambda - 2)(\lambda + 5) = 0$$

$\lambda = 2, -5$ are the eigen values of A

Consider $[A - \lambda I][x] = [0]$

$$\begin{bmatrix} (-19-\lambda) & 7 \\ -42 & (16-\lambda) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(-19-\lambda)x + 7y = 0$$

$$-42x + (16-\lambda)y = 0$$

Case 1 :- let $\lambda = 2$

$$-21x + 7y = 0 \quad \& \quad -42x + 14y = 0$$

$$y = 3x \quad \text{or} \quad \frac{y}{3} = \frac{x}{1}$$

$\therefore x_1 = (1, 3)^T$ is the eigen vector for $\lambda = 2$

Case 2 :- let $\lambda = -5$

$$-14x + 7y = 0 \quad \& \quad -42x + 21y = 0$$

$$y = 2x \quad \text{or} \quad \frac{y}{2} = \frac{x}{1}$$

$x_2 = (1, 2)^T$ is the eigen vector for $\lambda = -5$

Modal matrix $P = [x_1, x_2] = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$

$|P| = 2 - 3 = -1$

$P^{-1} = \frac{1}{|P|} \text{adj } P = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix}$

Now $P^{-1}AP = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$

$= \begin{bmatrix} -4 & 2 \\ -15 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$

$= \text{Diag}(2, -5)$

3. Diagonalize the matrix $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$

The ch. eq. is $|A - \lambda I| = 0$

$\begin{vmatrix} (-1-\lambda) & 2 \\ 2 & (-1-\lambda) \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} -(1+\lambda) & 2 \\ 2 & -(1+\lambda) \end{vmatrix} = 0$

$(1+\lambda)^2 - 4 = 0$

$(1+\lambda)^2 - 2^2 = 0$

$(1+\lambda+2)(1+\lambda-2) = 0$

$(\lambda+3)(\lambda-1) = 0$

$\lambda = -3, 1$ are the eigen values of A .

Consider $[A - \lambda I][x] = [0]$

$\begin{bmatrix} -(1+\lambda) & 2 \\ 2 & -(1+\lambda) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$-(1+\lambda)x + 2y = 0$$

$$2x - (1+\lambda)y = 0$$

Case 1 :- $\lambda = 1$

$$-2x + 2y = 0$$

$$x = y$$

$$\frac{x}{1} = \frac{y}{1}$$

$x_1 = (1, 1)'$ is the eigen vector for $\lambda = 1$

Case 2 :- $\lambda = -3$

$$2x + 2y = 0$$

$$x = -y$$

$$\frac{x}{1} = \frac{y}{-1}$$

$x_2 = (1, -1)'$ is the eigen vector for $\lambda = -3$

$$P = [x_1 \ x_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$|P| = -2, \quad P^{-1} = \frac{1}{|P|} \text{adj } P = \frac{1}{-2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\text{Now } P^{-1}AP = \frac{1}{-2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 3 \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 2 & 0 \\ 0 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$$

The ch. eqⁿ of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} (1-\lambda) & 1 \\ 3 & (-1-\lambda) \end{vmatrix} = 0$$

$$(1-\lambda)(-1-\lambda) - 3 = 0$$

$$\lambda^2 - 4 = 0$$

$\lambda = \pm 2$ are the eigen values of A

Consider $[A - \lambda I][x] = [0]$

$$\begin{bmatrix} (1-\lambda) & 1 \\ 3 & (-1-\lambda) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-\lambda)x + y = 0$$

$$3x - (1+\lambda)y = 0$$

Case 1 :- Let $\lambda = 2$

$$x = y$$

$$\frac{x}{1} = \frac{y}{1}$$

$x_1 = (1, 1)'$ is the eigen vector for $\lambda = 2$

Case 2 :- Let $\lambda = -2$

$$3x + y = 0$$

$$3x = -y$$

$$\frac{x}{1} = \frac{y}{-3}$$

$x_2 = (1, -3)'$ is the eigen vector for $\lambda = -2$

$$\text{Modal Matrix} = P = [x_1 \ x_2] = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$|P| = -4, \quad P^{-1} = \frac{1}{|P|} \text{Adj } P = \frac{1}{4} \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$$

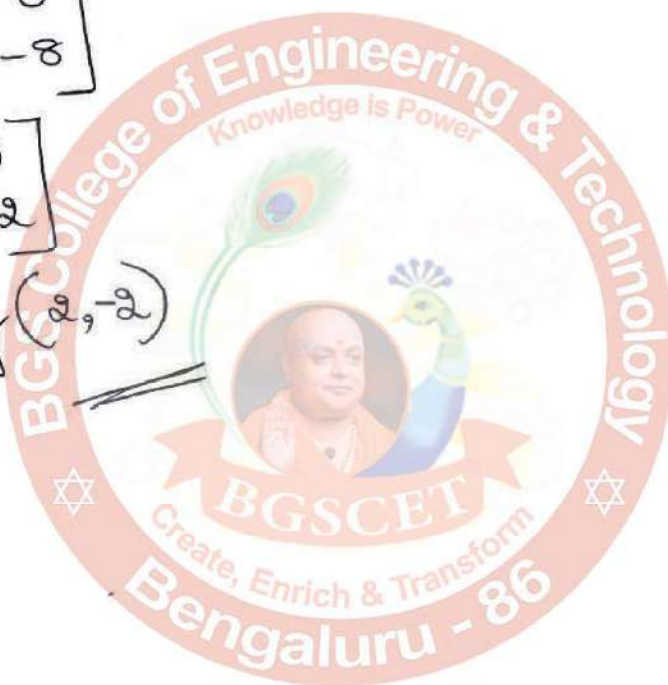
$$P^{-1}AP = \frac{1}{4} \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 2 & 6 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 8 & 0 \\ 0 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= \text{Diag}(2, -2)$$



Introduction :—

Group :— A non empty set equipped with one binary operation $*$ is called group if $*$ satisfies following Postulates (or axioms) .

1. Closure Property : $\forall a \text{ and } b \in G \Rightarrow a * b \in G$.
2. Associative Property : $\forall a, b, c \in G \Rightarrow a * (b * c) = (a * b) * c$
3. Existence of identity :

There exists an element $e \in G$ such that

$$a * e = a = e * a$$

The element e is called the identity .

4. Existence of inverse :

Each element of G possesses inverse, i.e $\forall a \in G \Rightarrow$ there exists an element $b \in G$ such that

$$a * b = e = b * a$$

Then b is called inverse of a and we write $b = a^{-1}$

$$\therefore a * a^{-1} = e = a^{-1} * a$$

* A group with commutative property is known as abelian group or commutative group .
i.e $\forall a, b \in G$, if $a * b = b * a$.

$\therefore$ ' $*$ ' is Commutative on G .

Eg:- i) $(I, +)$ is an abelian group as $(I, +)$ is a group and also $\forall a, b \in I$.

ii) $(\mathbb{Q}, \cdot)$, $(\mathbb{R}, \cdot)$, $(\mathbb{C}, \cdot)$ are examples of abelian group .

Rings:-

Suppose R is a non-empty set equipped with two binary operation called addition and multiplication and denoted by '+' and '•' respectively.

Then the algebraic structure $(R, +, \cdot)$ is known as ring if the following axioms are satisfied.

1) $(R, +)$ is an abelian group.

i) closure property for '+'

$$\forall a, b \in R \Rightarrow a+b \in R.$$

ii) Associative property for '+'

$$\forall a, b, c \in R \Rightarrow a+(b+c) = (a+b)+c$$

iii) Existence of identity element for '+'

$$\forall a \in R \exists 0 \in R \ni a+0 = a = 0+a.$$

0 is known as additive element or Zero element of the ring.

iv) Existence of inverse element for '+'.

$$\forall a \in R \exists -a \in R.$$

$$\text{such that } a+(-a) = 0 = -a+a.$$

$-a$ is known as additive inverse of a .

2) $(R, \cdot)$ is a semi group.

i) closure property for '•'

$$\forall a, b \in R \Rightarrow a \cdot b \in R.$$

ii) Associative property for '•'

$$\forall a, b, c \in R \Rightarrow a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

3) Multiplication '•' distributes over addition '+' from left and also from right.

$$\forall a, b, c \in R \quad \text{i) } a \cdot (b+c) = a \cdot b + a \cdot c$$

$$\text{ii) } (a+b) \cdot c = a \cdot c + b \cdot c$$

Eg:- $(\mathbb{I}, +, \cdot)$ is a ring.

* Commutative Ring:-

Ring $(R, +, \cdot)$ is a commutative ring,
if $\forall a, b \in R, a \cdot b = b \cdot a$.

Eg:- $(\mathbb{I}, +, \cdot)$.

* Ring With Unity:-

Ring $(R, +, \cdot)$ is known as ring with unity if
 $\forall a \in R \exists 1 \in R$ such that $a \cdot 1 = a = 1 \cdot a$.

* Commutative Ring with Unity:-

Ring $(R, +, \cdot)$ is commutative ring with unity,
if $\forall a, b \in R \Rightarrow a \cdot b = b \cdot a$ and $\forall a \in R \exists 1 \in R$
such that $a \cdot 1 = a = 1 \cdot a$.

Field:-

A commutative ring with unity in which every
non-zero element possesses their multiplicative inverse
is known as field.

i.e. $(F, +, \cdot)$ is a field if

(i) $(F, +)$ is an abelian group.

(ii) $(F, \cdot)$ is an abelian group.

(iii) Multiplication distributes over addition from the
left and also from the right.

Eg:- $(\mathbb{Q}, +, \cdot), (\mathbb{R}, +, \cdot), (\mathbb{C}, +, \cdot)$ are examples of fields.

* Note :-

If $(F, +, \cdot)$ is a field then F contains at least two elements, zero element and unit element i.e additive identity element and multiplicative identity element.



Vector Spaces:-

Let F be a field. A vector space over F , is a non-empty set V together with two operations (called addition and scalar multiplication) such that for each $u, v \in V$ there is a unique element $u+v \in V$ and for each $\alpha \in F$ and $u \in V$ there is a unique element $\alpha u \in V$, and it satisfies the following conditions:

- I. (i) $u+(v+w) = (u+v)+w$, for all $u, v, w \in V$
(Associativity)
- (ii) $u+v = v+u$, for all $u, v \in V$ (Commutativity)
- (iii) There exists an element $0 \in V$ such that
 $u+0 = u = 0+u$, for all $u \in V$ (Existence of identity element)
- (iv) For each $u \in V$, there exists a unique element $-u \in V$ such that $u+(-u) = 0 = (-u)+u$
(Existence of additive inverse)

II. There is an external composition in V over F called scalar multiplication.

$$\text{i.e. } \forall \alpha \in F \text{ and } u \in V \Rightarrow \alpha \cdot u \in V$$

In other words V is closed with respect to scalar multiplication.

III. The two compositions, i.e. vector addition and scalar multiplication satisfy the following postulates,

$\forall a, b \in F$ and $v, w \in V$

(i) $(a+b) \cdot v = a \cdot v + b \cdot v$

(ii) $a \cdot (b \cdot v) = (ab) \cdot v$

(iii) $a \cdot (v+w) = a \cdot v + a \cdot w$

(iv) $1 \cdot (v) = v$.

* Vectors in R^n :-

The set of all ordered triples (a, b, c) of real numbers is called Euclidean 3-space and is denoted by R^3 & the set of all n -tuples of real numbers, denoted by R^n , is called Euclidean n -space.

Eg:- The ordered pair $(2, -3)$ belongs to R^2 ; it is a 2-tuple of dimension two.

The ordered triple $(7, 3, 6)$ belongs to R^3 ; it is a 3-tuple of dimension three.

Examples or Problems :-

1) Prove that the set of all vectors in a plane over the field of real numbers is a vector space with respect to vector addition and scalar multiplication.

Solⁿ:- Let V denotes the set of all coplanar vectors and R be the field of real numbers.

$\therefore$ The elements of V are the ordered pairs (x, y) where $x, y \in R$.

$$V = \{(x, y) : x, y \in R\}$$

I) $(V, +)$ is an abelian group.

i) Associativity:-

We know that for all $u, v, w \in V$
$$(u+v)+w = u+(v+w)$$

ii) Commutativity:-

We know that for all $u, v \in V$
$$u+v = v+u$$

iii) Existence of additive identity:-

For every vector $u \in V$ there exists a zero vector $0 \in V$ such that $u+0 = 0+u = u$.

iv) Existence of additive inverse:-

For every vector $u \in V$ there exists a vector $-u \in V$ such that $u+(-u) = (-u)+u = 0$.

Thus V is an abelian group with respect to vector addition.

II) Scalar multiplication in V .

i) For $u, v \in V$ and $\alpha \in \mathbb{R}$ we have

$$\alpha(u+v) = \alpha u + \alpha v$$

ii) For $u \in V$ and $a, b \in \mathbb{R}$, we have

$$(a+b)u = au + bu$$

iii) For $u \in V$ and $a, b \in \mathbb{R}$, we have

$$a(bu) = (ab)u$$

iv) For $u \in V$ and $1 \in \mathbb{R}$, we have

$$1u = u$$

Thus set V of coplanar vectors satisfies all the properties of vector addition and scalar multiplication.
 $\therefore V$ is a vector space.

2) Prove that the set C of all complex numbers (i.e. the set of all ordered pairs of real numbers) is a vector space over the field $\mathbb{R}$ of all real numbers where vector addition is defined by

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2), \text{ for all } (x_1, x_2), (y_1, y_2) \in C$$

and scalar multiplication is defined by

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2), \text{ for all } \alpha \in \mathbb{R}.$$

Solⁿ:- We observe that G is closed under vector addition and under scalar multiplication.

I. $(C, +)$ is an abelian group.

(i) Associativity:

For all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in G$, we have

$$\begin{aligned} (x_1, x_2) + [(y_1, y_2) + (z_1, z_2)] &= (x_1, x_2) + (y_1 + z_1, y_2 + z_2) \\ &= (x_1 + y_1 + z_1, x_2 + y_2 + z_2) \\ &= (x_1 + y_1, x_2 + y_2) + (z_1, z_2) \\ &= [(x_1, x_2) + (y_1, y_2)] + (z_1, z_2) \end{aligned}$$

(ii) Commutativity:

For all $(x_1, x_2), (y_1, y_2) \in G$, we have

$$\begin{aligned} (x_1, x_2) + (y_1, y_2) &= (x_1 + y_1, x_2 + y_2) \\ &= (y_1 + x_1, y_2 + x_2) \\ &= (y_1, y_2) + (x_1, x_2) \end{aligned}$$

(iii) Existence of identity:

For $(x_1, x_2) \in G$, there exists $(0, 0) \in G$ such that

$$\begin{aligned} (x_1, x_2) + (0, 0) &= (x_1 + 0, x_2 + 0) \\ &= (x_1, x_2) \end{aligned}$$

$$(0 + 0) + (x_1, x_2) = (0 + x_1, 0 + x_2) = (x_1, x_2)$$

$$\therefore (x_1, x_2) + (0, 0) = (x_1, x_2) = (0, 0) + (x_1, x_2),$$

(iv) Existence of inverse :

For any $(x_1, x_2) \in G$, there exists $(-x_1, -x_2) \in G$ such that

$$(x_1, x_2) + (-x_1, -x_2) = (0, 0) = (-x_1, -x_2) + (x_1, x_2).$$

Thus C is an abelian group with respect to vector addition.

II. Properties of Scalar multiplication in C .

$$\begin{aligned} \text{(i)} \quad \alpha [(x_1, x_2) + (y_1, y_2)] &= \alpha (x_1 + y_1, x_2 + y_2) \\ &= (\alpha(x_1 + y_1), \alpha(x_2 + y_2)) \\ &= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2) \\ &= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2) \\ &= \alpha(x_1, x_2) + \alpha(y_1, y_2) \end{aligned}$$

Thus $\alpha [(x_1, x_2) + (y_1, y_2)] = \alpha(x_1, x_2) + \alpha(y_1, y_2)$
for all $(x_1, x_2), (y_1, y_2) \in G, \alpha \in R$.

$$\begin{aligned} \text{(ii)} \quad (a+b)(x_1, x_2) &= ((a+b)x_1, (a+b)x_2) \\ &= (ax_1 + bx_1, ax_2 + bx_2) \\ &= (ax_1, ax_2) + (bx_1, bx_2) \\ &= a(x_1, x_2) + b(x_1, x_2) \end{aligned}$$

Thus $(a+b)(x_1, x_2) = a(x_1, x_2) + b(x_1, x_2)$ for all
 $(x_1, x_2), (y_1, y_2) \in G$ & $a, b \in R$.

$$\begin{aligned} \text{(iii)} \quad a(b(x_1, x_2)) &= a(bx_1, bx_2) = (abx_1, abx_2) \\ a(b(x_1, x_2)) &= (ab)(x_1, x_2) \end{aligned}$$

$$\text{(iv)} \quad 1.(x_1, x_2) = (x_1, x_2) \text{ for all } (x_1, x_2) \in G$$

Thus the set G satisfies all the properties of vector space.

Hence G is a vector space over R .

3) Show that the set V of all real valued continuous functions of x defined on interval $[0, 1]$ is a vector space over the field R of real numbers with respect to vector addition and scalar multiplication defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x), \text{ for all } f_1, f_2 \in V$$

$$(\alpha f_1)(x) = \alpha f_1(x), \text{ for all } \alpha \in R, f_1 \in V.$$

Solⁿ:-

I. $(V, +)$ is an abelian group.

(i) Associativity:

Let $f_1, f_2, f_3 \in V$ be arbitrary.

$$\begin{aligned} [(f_1 + f_2) + f_3](x) &= (f_1 + f_2)(x) + f_3(x) \\ &= [f_1(x) + f_2(x)] + f_3(x) \\ &= f_1(x) + [f_2(x) + f_3(x)] \\ &= f_1(x) + (f_2 + f_3)(x) \\ &= [f_1 + (f_2 + f_3)](x) \end{aligned}$$

$$(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3).$$

(ii) Commutativity:

Let $f_1, f_2 \in V$ be arbitrary.

$$\begin{aligned} (f_1 + f_2)(x) &= f_1(x) + f_2(x) \\ &= f_2(x) + f_1(x) \\ &= (f_2 + f_1)(x), \text{ for all } x. \end{aligned}$$

$$f_1 + f_2 = f_2 + f_1.$$

(iii) Existence of Identity:

Define a function 0 such that $0(x) = 0$ (real number), for all $x \in [0, 1]$.

Also, 0 is a continuous function and belongs to V

$$(0 + f_1)(x) = 0(x) + f_1(x) = 0 + f_1(x) = f_1(x)$$

$$(f_1 + 0)(x) = f_1(x) + 0(x) = f_1(x)$$

$$0 + f_1 = f_1 + 0 = f_1$$

Thus, the function 0 defined above is an additive identity element of V .

(iv) Existence of Inverse:

For a function f_1 , the function $-f_1$ defined by $(-f_1)(x) = -f_1(x)$, is called additive

$$\text{inverse, as } [f_1 + (-f_1)](x) = f_1(x) + (-f_1)(x) \\ = f_1(x) - f_1(x) = 0 = 0(x)$$

$$\text{Similarly, } [-f_1 + f_1](x) = 0(x).$$

Thus the set V is an abelian group under addition.

II. Properties of Scalar Multiplication in V .

(i) For $f_1, f_2 \in V$ and $\alpha \in R$, we have

$$\begin{aligned} [\alpha(f_1 + f_2)]x &= \alpha[(f_1 + f_2)(x)] \\ &= \alpha[f_1(x) + f_2(x)] \\ &= \alpha f_1(x) + \alpha f_2(x) \\ &= (\alpha f_1)x + (\alpha f_2)x \\ &= (\alpha f_1 + \alpha f_2)x \end{aligned}$$

$$\alpha(f_1 + f_2) = \alpha f_1 + \alpha f_2$$

(ii) For $f_1 \in V$ and $a, b \in R$.

$$\begin{aligned} [(a+b)f_1](x) &= (a+b)f_1(x) \\ &= a f_1(x) + b f_1(x) \\ &= (a f_1)x + (b f_1)x \\ &= (a f_1 + b f_1)(x) \end{aligned}$$

$$(a+b)f_1 = a f_1 + b f_1$$

(iii) For $f_1 \in V$ and $a, b \in R$,

$$\begin{aligned} [a(b f_1)]x &= a[(b f_1)x] \\ &= a[b f_1(x)] \\ &= (ab)f_1(x) \\ &= [(ab)f_1]x \end{aligned}$$

$$a(b f_1) = (ab)f_1$$

(iv) For $f_1 \in V$ and $1 \in R$, we have

$$(1, f_1)x = 1 \cdot f_1(x) = f_1(x)$$

$$1 \cdot f_1 = f_1$$

$\therefore$ Thus V satisfies all the properties of a vector space and hence V is a vector space.

4) Show that the set of all matrices of the type $m \times n$ where m and n are fixed positive integers is a vector space over R with respect to matrix addition and multiplication of matrix by a scalar (i.e., a real number)

Solⁿ: Let M denotes the set of all matrices of type $m \times n$.

I. $(M, +)$ is an abelian group.

(i) Associativity:

Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$, $C = [c_{ij}]_{m \times n}$ be three matrices belonging to set M .

$$\begin{aligned}(A+B)+C &= ([a_{ij}] + [b_{ij}]) + [c_{ij}] \\&= [a_{ij} + b_{ij}] + [c_{ij}] \\&= [(a_{ij} + b_{ij}) + c_{ij}] \\&= [a_{ij} + (b_{ij} + c_{ij})] \\&= [a_{ij}] + [b_{ij} + c_{ij}] \\&= [a_{ij}] + ([b_{ij}] + [c_{ij}]) \\&= A + (B+C).\end{aligned}$$

(ii) Commutativity:

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two matrices belonging to set M .

$$\begin{aligned}
 A+B &= [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] \\
 &= [b_{ij} + a_{ij}] \\
 &= [b_{ij}]_{m \times n} + [a_{ij}]_{m \times n} \\
 &= B + A.
 \end{aligned}$$

(iii) Existence of Identity :

Let $A = [a_{ij}]_{m \times n} \in M$ be arbitrary.

We know that a zero matrix (null matrix) of type $m \times n$ also belongs to the set M and is denoted by O .

$$\text{Now, } A + O = [a_{ij}] + [0] = [a_{ij} + 0] = [a_{ij}] = A$$

$$\text{Similarly, } O + A = A.$$

(iv) Existence of Inverse :

If $A = [a_{ij}]_{m \times n}$ belongs to the set M , then

$-A = [-a_{ij}]_{m \times n}$ also belongs to the set M .

$$\begin{aligned}
 \text{Now, } A + (-A) &= [a_{ij}] + [-a_{ij}] = [a_{ij} - a_{ij}] \\
 &= [0] = O
 \end{aligned}$$

$$\text{Similarly, } (-A) + A = O$$

Thus the set M is an abelian group under addition.

II. Properties of Scalar multiplication in M .

(i) Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n} \in M$ and $\alpha \in R$, then

$$\alpha(A+B) = \alpha([a_{ij}] + [b_{ij}])$$

$$= \alpha[a_{ij} + b_{ij}]$$

$$= [\alpha(a_{ij} + b_{ij})]$$

$$= [\alpha \cdot a_{ij} + \alpha \cdot b_{ij}] \quad [\because \text{Multiplication is distributive in } R]$$

$$= [\alpha \cdot a_{ij}] + [\alpha \cdot b_{ij}]$$

$$= \alpha[a_{ij}] + \alpha[b_{ij}] = \alpha A + \alpha B.$$

(ii) Let $A = [a_{ij}]_{m \times n} \in M$ and $a, b \in R$, then

$$\begin{aligned}(a+b)A &= (a+b)[a_{ij}] \\&= [(a+b)a_{ij}] \\&= [a \cdot a_{ij} + b \cdot a_{ij}] \\&= a[a_{ij}] + b[a_{ij}] \\&= aA + bA.\end{aligned}$$

(iii) Let $A = [a_{ij}]_{m \times n} \in M$ and $a, b \in R$ then

$$\begin{aligned}a(bA) &= a(b[a_{ij}]) \\&= a([ba_{ij}]) \\&= [(ab)a_{ij}] \\&= (ab)[a_{ij}] \\&= (ab)A \\a(bA) &= (ab)A.\end{aligned}$$

(iv) Let $A = [a_{ij}]_{m \times n} \in M$, then

$$1 \cdot A = 1 \cdot [a_{ij}] = [1 \cdot a_{ij}] = [a_{ij}] = A.$$

Thus M satisfies all the properties of vector space and hence M is a vector space over R .

5) If V is an abelian group of positive real numbers for multiplication. Define scalar multiplication in V by $a \cdot x = x^a$ where $a \in R$, $x \in V$. Show that V is a vector space over R .

Solⁿ: - Since $V = R^+$ is an abelian group for multiplication.

There are two operations

(i) $\times$, i.e multiplication in R^+

(ii) $\cdot$, i.e Scalar multiplication in R^+

$$a \in \mathbb{R}, x \in \mathbb{R}^+$$

$$a \cdot x = x^a$$

and $x^a \in \mathbb{R}^+$ as $x \in \mathbb{R}^+$, x^a is always positive.

Eg:- Let $a = -3$
 $-3 \cdot x = x^{-3} = \frac{1}{x^3}$ is a positive real number.

Hence x^{-3} is an element of $\mathbb{R}^+$.

Now,

I. $(V, \times)$ is an abelian group.

II. $\forall a \in \mathbb{R}, x \in V \Rightarrow a \cdot x = x^a \in V$.

III. (i) $(a+b) \cdot x = x^{a+b} = x^a \times x^b = (a \cdot x) \times (b \cdot x)$

$$\begin{aligned} \text{(ii)} \quad a \cdot (x_1 \times x_2) &= (x_1 \times x_2)^a \\ &= x_1^a \times x_2^a \\ &= (a \cdot x_1) \times (a \cdot x_2) \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad a \cdot (b \cdot x) &= (b \cdot x)^a \\ &= (x^b)^a = x^{ba} \end{aligned}$$

$$= x^{ab} \quad (\because \mathbb{R} \text{ is commutative for multiplication})$$

$$= ab \cdot (x)$$

$$\text{(iv)} \quad 1 \cdot (x) = x' = x$$

Hence $(V, \times, \bullet)$ is a vector space.

Subspaces :-

A non-empty subset W of a vector space $V(F)$ is said to form a subspace of V if W is also a vector space over F with the same addition and scalar multiplication as for V .

Eg:- Let $W_1 = \{(a, 0, 0) : a \in \mathbb{R}\}$
 $W_2 = \{(a, b, 0) : a, b \in \mathbb{R}\}$.

Here W_1 is a subspace of W_2 . Also W_1 and W_2 are subspace of $\mathbb{R}^3$.

* Necessary and Sufficient conditions for a subspace:
(Only statement, no proof).

Theorem 1: W is a subspace of $V(F)$ iff

- i) W is non empty.
- ii) W is closed under vector addition.
i.e. $\forall w_1, w_2 \in W \Rightarrow w_1 + w_2 \in W$.
- iii) W is closed under scalar multiplication.
i.e. $\forall a \in F$ and $w \in W \Rightarrow a.w \in W$.

Theorem 2: W is a subspace of $V(F)$ iff

- i) W is non empty.
- ii) $\forall a, b \in F$ and $v, w \in W \Rightarrow$
 $a.v + b.w \in W$.

Problems:-

1) Show that W is a subspace of $V(R)$ where $W = \{f: f(9) = 0\}$

Solⁿ:- Since $0 \in W$ as $0(9) = 0$
So W is non empty set.

Let $f, g \in W$.

i.e. $f(9) = 0$ and $g(9) = 0$

then $\forall a, b \in R$

$$(a.f + b.g)(9) = a.f(9) + b.g(9) = a.0 + b.0 = 0$$

Hence $a.f + b.g \in W$.

By theorem (2), W is a subspace of $V(R)$.

2) Show that W is a subspace of $V(R)$ where $W = \{f: f(2) = f(1)\}$

Solⁿ:- $0 \in W$ since $0(2) = 0 = 0(1)$
hence W is a non empty set.

Let $f, g \in W$ then $f(2) = f(1)$ & $g(2) = g(1)$

then $\forall a, b \in R$.

$$\begin{aligned}(a.f + b.g)(2) &= a.f(2) + b.g(2) = a.f(1) + b.g(1) \\ &= (a.f + b.g)(1)\end{aligned}$$

hence $a.f + b.g \in W$.

So by theorem 2, W is a subspace of $V(R)$.

3) Let $V = R^3$ be the Euclidean 3-space. Let

$W = \{(x, y, z): ax + by + cz = 0; x, y, z \in R\}$, a, b, c being real numbers. Show that W is a subspace of V .

(OR)

Show that any plane passing through the origin is a subspace of $\mathbb{R}^3$.

Solⁿ:- Let $u = (x_1, y_1, z_1)$, $v = (x_2, y_2, z_2)$ be any two elements of W where $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$.

Then $ax_1 + by_1 + cz_1 = 0$ and $ax_2 + by_2 + cz_2 = 0$.

For $\alpha \in \mathbb{R}$, $\alpha u + v = \alpha(x_1, y_1, z_1) + (x_2, y_2, z_2)$

$$\alpha u + v = (\alpha x_1, \alpha y_1, \alpha z_1) + (x_2, y_2, z_2)$$

$$= (\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + z_2) \rightarrow (1)$$

Where $\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + z_2 \in \mathbb{R}$.

Now

$$a(\alpha x_1 + x_2) + b(\alpha y_1 + y_2) + c(\alpha z_1 + z_2) =$$

$$\alpha(ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2)$$

$$= \alpha(0) + 0 = 0 \rightarrow (2)$$

From (1) and (2), we have

$$\alpha u + v \in W.$$

Thus, W is a subspace of V .

4) Let V be the vector space of all square matrices over $\mathbb{R}$.

Determine which of the following are subspaces of V .

$$(i) W = \left\{ \begin{bmatrix} x & y \\ z & 0 \end{bmatrix} : x, y, z \in \mathbb{R} \right\} \quad (ii) W = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

$$(iii) W = \{A : A \in V \text{ and } A \text{ is singular}\} \quad (iv) W = \{A : A \in V, A^2 = A\}$$

Solⁿ:- (i) Let $A = \begin{bmatrix} x_1 & y_1 \\ x_1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} x_2 & y_2 \\ x_2 & 0 \end{bmatrix}$ be any two elements of W .

If $a, b \in \mathbb{R}$ then

$$aA + bB = a \begin{bmatrix} x_1 & y_1 \\ x_1 & 0 \end{bmatrix} + b \begin{bmatrix} x_2 & y_2 \\ x_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 & ay_1 \\ ax_1 & 0 \end{bmatrix} + \begin{bmatrix} bx_2 & by_2 \\ bx_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + bx_2 & ay_1 + by_2 \\ ax_1 + bx_2 & 0 \end{bmatrix}$$

which is a matrix of the type $\begin{bmatrix} x & y \\ x & 0 \end{bmatrix}$ and

$$ax_1 + bx_2, ay_1 + by_2, ax_1 + bx_2 \in \mathbb{R}.$$

$$\therefore aA + bB \in W.$$

Thus, W is a sub-space of V .

ii) Let $A = \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix}$, $B = \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix}$ be any two elements

of W .

If $a, b \in \mathbb{R}$, then

$$aA + bB = a \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix} + b \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + bx_2 & 0 \\ 0 & ay_1 + by_2 \end{bmatrix}$$

which is a matrix of the type $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ and

$$ax_1 + bx_2, ay_1 + by_2 \in \mathbb{R}.$$

$$\therefore aA + bB \in W.$$

Thus W is a subspace of V .

(iii) Here W is the set of singular matrices.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ be two square matrices.

Since $|A|=0$ and $|B|=0$. therefore $A, B \in W$.

If $a, b \in \mathbb{R}$ are non zero, then

$$\begin{aligned} aA + bB &= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \\ &= \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \end{aligned}$$

Also, $|aA + bB| = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab \neq 0$, as none of a and b is zero.

$\therefore aA + bB \notin W$.

Thus, W is not a sub-space of V .

(iv) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ so that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$

$\therefore A \in W$.

$$\text{Now } A + A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(A+A)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= A + A$$

$\therefore (A+A)$ is not a member of the set W .

$\Rightarrow W$ is not closed under addition.

Hence W is not a sub-space of V .

5) Let V be the vector space of all real valued continuous functions over $\mathbb{R}$. Show that the set W of solutions of differential equations $5 \frac{d^2y}{dx^2} - 7 \frac{dy}{dx} + 2y = 0$ is a subspace of V .

Solⁿ: - Let $y_1, y_2 \in W$. Then y_1, y_2 are solutions of differential equations.

$$5 \frac{d^2y}{dx^2} - 7 \frac{dy}{dx} + 2y = 0 \rightarrow (1)$$

i.e. $5 \frac{d^2y_1}{dx^2} - 7 \frac{dy_1}{dx} + 2y_1 = 0$; $5 \frac{d^2y_2}{dx^2} - 7 \frac{dy_2}{dx} + 2y_2 = 0$.

Let $a, b \in \mathbb{R}$ be arbitrary.

$$5 \frac{d^2}{dx^2} (ay_1 + by_2) - 7 \frac{d}{dx} (ay_1 + by_2) + 2(ay_1 + by_2)$$

$$= 5 \frac{d^2}{dx^2} (ay_1) - 7 \frac{d}{dx} (ay_1) + 2(ay_1) +$$

$$5 \frac{d^2}{dx^2} (by_2) - 7 \frac{d}{dx} (by_2) + 2(by_2).$$

$$= a \left[5 \frac{d^2y_1}{dx^2} - 7 \frac{dy_1}{dx} + 2y_1 \right] + b \left[5 \frac{d^2y_2}{dx^2} - 7 \frac{dy_2}{dx} + 2y_2 \right]$$

$$= a \cdot 0 + b \cdot 0 = 0.$$

Thus $ay_1 + by_2$ is a solution of (1) & so $ay_1 + by_2 \in W$.

$\therefore W$ is a subspace of V .

Linear Combination :-

Let V be a vector space over a field F and let $v_1, v_2, \dots, v_n \in V$.

Any vector of the form $a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n$ in V , where $a_i \in F$ is called a linear combination of $v_1, v_2, \dots, v_n$.

Linear Dependence :-

Let V be a vector space over the field F . The vectors $v_1, v_2, \dots, v_n$ are said to be linearly dependent over F , if $\exists$ scalars $a_1, a_2, \dots, a_n \in F$ not all zero but linear combination is zero.

$$\text{i.e. } a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n = 0.$$

but all $a_i \neq 0$, where $i \in \mathbb{N}$.

Linear Independence :-

Let V be a vector space over the field F . The vectors $v_1, v_2, \dots, v_n$ are said to be linearly independent over F , if $\exists$ scalars $a_1, a_2, \dots, a_n \in F$ such that

$$a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n = 0 \Rightarrow \text{all } a_i = 0 \text{ where } i \in \mathbb{N}.$$

Linear Span :-

Let S be a subset of the vector space V over the field F . The set of all linear combinations of vectors in S is called a linear span of S and is denoted by $L(S)$.

If $S = \phi$, then $L(S) = 0$.

Basis or Base of Vector space V :-

Let V be a vector space over the field F . The set of vectors $\{v_1, v_2, \dots, v_n\}$ is called a basis of V , if

- (i) $v_1, v_2, \dots, v_n$ are linearly independent,
- (ii) $v_1, v_2, \dots, v_n$ span V . i.e. each vector of V can be uniquely expressed as linear combination of $v_1, v_2, \dots, v_n$.

Dimension of a vector space V :-

Number of elements in a basis of vector space V is called the dimension of V and is denoted by $\dim V$.

If V contains a basis with n elements then the $\dim V = n$.

Note :-

- 1) The vector space $\{0\}$ is defined to have $\dim 0$, since empty set ϕ is independent and generates $\{0\}$.

$$\begin{aligned}\therefore \dim \{0\} &= \text{Number of elements in } \phi \\ &= 0 \text{ [Since no element is in } \phi \text{]}\end{aligned}$$

- 2) When a vector space is not of finite dimension, it is said to be of infinite dimension.

Problems :-

1) Express the vector $v = (1, -2, 5)$ as a linear combination of the vectors $v_1 = (1, 1, 1)$, $v_2 = (1, 2, 3)$, $v_3 = (2, -1, 1)$ in the vector space $R^3(R)$.

Soln:- Let $v = a_1 v_1 + a_2 v_2 + a_3 v_3$; $a_1, a_2, a_3 \in R$

$$(1, -2, 5) = a_1(1, 1, 1) + a_2(1, 2, 3) + a_3(2, -1, 1)$$

$$(1, -2, 5) = (a_1 + a_2 + 2a_3, a_1 + 2a_2 - a_3, a_1 + 3a_2 + a_3)$$

Equating the corresponding elements, we get

$$a_1 + a_2 + 2a_3 = 1; a_1 + 2a_2 - a_3 = -2; a_1 + 3a_2 + a_3 = 5$$

Solving above equations, we get

$$a_1 = -6, a_2 = 3, a_3 = 2$$

$$\text{Hence } (1, -2, 5) = -6(1, 1, 1) + 3(1, 2, 3) + 2(2, -1, 1).$$

2) Write the vector $v = (1, 3, 9)$ as a linear combination of the vectors $u_1 = (2, 1, 3)$, $u_2 = (1, -1, 1)$, $u_3 = (3, 1, 5)$

(Another method of solving)

Soln:- Let $v = x u_1 + y u_2 + z u_3$

$$(1, 3, 9) = x(2, 1, 3) + y(1, -1, 1) + z(3, 1, 5)$$

$$\Rightarrow 2x + y + 3z = 1; x - y + z = 3; 3x + y + 5z = 9$$

Consider $AX = B$.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & 5 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

Augmented matrix,

$$[A:B] = \begin{bmatrix} 2 & 1 & 3 & : & 1 \\ 1 & -1 & 1 & : & 3 \\ 3 & 1 & 5 & : & 9 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2$$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 2 & 1 & 3 & : & 1 \\ 3 & 1 & 5 & : & 9 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - 3R_1$$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 3 & 1 & : & -5 \\ 0 & 4 & 2 & : & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 3 & 1 & : & -5 \\ 0 & 1 & 1 & : & 5 \end{bmatrix}$$

$$R_3 \leftrightarrow R_2$$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 1 & 1 & : & 5 \\ 0 & 3 & 1 & : & -5 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 3R_2$$

$$[A:B] = \begin{bmatrix} 1 & -1 & 1 & : & 3 \\ 0 & 1 & 1 & : & 5 \\ 0 & 0 & -2 & : & -20 \end{bmatrix}$$

$\therefore \text{Rank } [A] = \text{Rank } [A:B] = 3 = \text{Number of unknowns}$

$\therefore$ The system of linear equations is consistent and possesses a unique soln.

$$x + y + z = 3; \quad y + z = 5; \quad -2z = -20$$

$$\therefore z = 10, \quad y = -5, \quad x = -12$$

$$\therefore (1, 3, 9) = -12(2, 1, 3) - 5(1, -1, 1) + 10(3, 1, 5).$$

3) Write the vector $v = (4, 2, 1)$ as a linear combination of the vectors $u_1 = (1, -3, 1)$, $u_2 = (0, 1, 2)$, $u_3 = (5, 1, 37)$.

Solⁿ:- Let $v = x u_1 + y u_2 + z u_3$

$$(4, 2, 1) = x(1, -3, 1) + y(0, 1, 2) + z(5, 1, 37)$$

$$\Rightarrow x + 0y + 5z = 4$$

$$-3x + y + z = 2$$

$$x + 2y + 37z = 1$$

$$\text{Let } AX = B. \text{ Where } A = \begin{bmatrix} 1 & 0 & 5 \\ -3 & 1 & 1 \\ 1 & 2 & 37 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{Consider, } [A:B] = \left[\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ -3 & 1 & 1 & 2 \\ 1 & 2 & 37 & 1 \end{array} \right]$$

$$R_2 \rightarrow R_2 + 3R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & 16 & 14 \\ 0 & 2 & 32 & -3 \end{array} \right]$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & 16 & 14 \\ 0 & 0 & 0 & -31 \end{array} \right]$$

$$\text{Rank}[A] = 2 \neq \text{Rank}[A:B] = 3.$$

Since $\text{Rank}[A] \neq \text{Rank}[A:B]$. Hence the system of linear equations is inconsistent.

i.e. solⁿ does not exist.

$\therefore v$ cannot be expressed as linear combination of the vectors u_1, u_2, u_3 .

4) For what value of k (if any) the vector $v = (1, -2, k)$ can be expressed as a linear combination of vectors $v_1 = (3, 0, -2)$ and $v_2 = (2, -1, -5)$ in $R^3(R)$.

Sol:- Since vector $v = (1, -2, k)$ is a linear combination of $v_1 = (3, 0, -2)$ and $v_2 = (2, -1, -5)$; therefore there exist scalars a and b such that

$$v = av_1 + bv_2$$

$$(1, -2, k) = a(3, 0, -2) + b(2, -1, -5)$$

$$(1, -2, k) = (3a + 2b, -b, -2a - 5b)$$

Equating the corresponding elements,

$$3a + 2b = 1, \quad -b = -2, \quad -2a - 5b = k$$

$$3a + 4 = 1 \quad b = 2 \quad -2(-1) - 5(2) = k$$

$$a = -1$$

$$2 - 10 = k$$

$$\therefore k = -8$$

5) Find a condition on a, b, c so that $w = (a, b, c)$ is a linear combination of $u = (1, -3, 2)$ and $v = (2, -1, 1)$ in R^3 so that $w \in \text{span}(u, v)$.

Sol:- Let $w = xu + yv$

$$(a, b, c) = x(1, -3, 2) + y(2, -1, 1)$$

$$x + 2y = a; \quad -3x - y = b; \quad 2x + y = c$$

Let $AX = B$.

$$\text{Consider } [A:B] = \begin{bmatrix} 1 & 2 & : & a \\ -3 & -1 & : & b \\ 2 & 1 & : & c \end{bmatrix}$$

$$R_2 \rightarrow R_2 + 3R_1$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$[A:B] = \begin{bmatrix} 1 & 2 & : & a \\ 0 & 5 & : & b+3a \\ 0 & -3 & : & c-2a \end{bmatrix}$$

$$R_2 \rightarrow \frac{R_2}{5}$$

$$[A:B] = \begin{bmatrix} 1 & 2 & : & a \\ 0 & 1 & : & \frac{b+3a}{5} \\ 0 & -3 & : & c-2a \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 3R_2$$

$$[A:B] = \begin{bmatrix} 1 & 2 & : & a \\ 0 & 1 & : & \frac{b+3a}{5} \\ 0 & 0 & : & \frac{5c-a+3b}{5} \end{bmatrix}$$

$$\text{Rank } [A] = 2$$

The system of linear equations will be consistent if
 $\text{Rank } [A:B] = 2$

$$\text{So, } \frac{5c-a+3b}{5} = 0 \Rightarrow a-3b-5c=0$$

i.e. w is linear combination of u and v iff

$$a-3b-5c=0.$$

6) Express the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix}$ in the vector space of 2×2 matrices as a linear combination of

$$B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Soln:- Let $A = a_1 B + a_2 C + a_3 D$; $a_1, a_2, a_3 \in \mathbb{R} \rightarrow (1)$

$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = a_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 + a_3 & a_1 + a_2 - a_3 \\ -a_2 & -a_1 \end{bmatrix}$$

Equating the corresponding elements,

$$a_1 + a_2 + a_3 = 3 ; a_1 + a_2 - a_3 = -1 ; -a_2 = 1 ; -a_1 = -2$$

Solving the above equations,

$$a_1 = 2, a_2 = -1, a_3 = 2$$

$\therefore$ (1) $\Rightarrow$

$$A = 2B - C - 2D$$

or
$$\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} - 2 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

which is the required linear combination of A.

7) Express the matrix $\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix}$ as a linear combination of the matrices $A = \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$.

Soln:- Let $\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix} = a_1 \begin{bmatrix} 0 & -3 \\ 2 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} + a_3 \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$;
 $a_1, a_2, a_3 \in \mathbb{R}$.

$$\begin{bmatrix} 2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 0 + 0 + 2a_3 & -3a_1 + 3a_3 \\ 2a_1 + 2a_2 & a_2 + 5a_3 \end{bmatrix}$$

Equating the corresponding elements $2 = 2a_3 \rightarrow$ (1)
 $0 = -3a_1 + 3a_3 \rightarrow$ (2)

$$2 = 2a_3 \Rightarrow a_3 = 1$$

$$4 = 2a_1 + 2a_2 \rightarrow$$
 (3)

$$0 = -3a_1 + 3a_3 \Rightarrow a_1 = 1$$

$$-5 = a_2 + 5a_3 \rightarrow$$
 (4)

$$4 = 2a_1 + 2a_2 \Rightarrow a_2 = 1$$

But $a_1 = 1, a_2 = 1, a_3 = 1$ do not satisfy eqⁿ (4)

Thus equations (1), (2) & (3) have no solution.

Hence, the given matrix cannot be expressed as a linear combination of A, B, C.

Note:-

* Two vectors v_1 and v_2 are linearly dependent iff one of them is a multiple of the other.

Eg:- Determine whether or not the vectors v_1, v_2 are linearly dependent.

i) $v_1 = (1, 3, 9)$ $v_2 = (2, 4, 1)$.

→ No vector is the multiple of the other hence v_1 & v_2 are not linearly dependent.

ii) $v_1 = (3, 4)$ $v_2 = (6, 8)$.

→ $v_2 = 2v_1$, hence v_1 and v_2 are linearly dependent vectors.

8) Determine whether the vectors $v_1 = (1, 2, 3)$ $v_2 = (3, 1, 7)$ and $v_3 = (2, 5, 8)$ are linearly dependent or linearly independent.

Soln:- $xv_1 + yv_2 + zv_3 = 0$

$$x(1, 2, 3) + y(3, 1, 7) + z(2, 5, 8) = (0, 0, 0)$$

$$x + 3y + 2z = 0; \quad 2x + y + 5z = 0; \quad 3x + 7y + 8z = 0.$$

Let $AX = 0$ where $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 5 \\ 3 & 7 & 8 \end{bmatrix}$

$$\begin{aligned} R_2 &\rightarrow R_2 - 2R_1 \\ R_3 &\rightarrow R_3 - 3R_1 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 1 \\ 0 & -2 & 2 \end{bmatrix} \begin{aligned} R_2 &\rightarrow -R_2 \\ R_3 &\rightarrow -\frac{R_3}{2} \end{aligned}$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & -1 \\ 0 & 1 & -1 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 5 & -1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 5R_2$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

$\text{Rank}[A] = 3 = \text{Number of unknowns}.$

$\Rightarrow$ Homogeneous system of linear equations possesses unique solution.

$$\text{i.e. } x=0, y=0, z=0.$$

So vectors v_1, v_2, v_3 are linearly independent.

9) Determine whether the vectors $v_1 = (1, 9, 3)$, $v_2 = (2, 5, 4)$, $v_3 = (0, 0, 0)$ are linearly dependent or linearly independent.

Soln:- vector v_1, v_2, v_3 are linearly dependent as linearly independent set of vectors can not contain zero vector.

10) Determine whether the vectors $v_1 = (1, 4, 9)$, $v_2 = (3, 1, 4)$, $v_3 = (9, 3, 12)$ are linearly dependent or linearly independent.

$$\text{Soln:- } x v_1 + y v_2 + z v_3 = 0$$

$$x(1, 4, 9) + y(3, 1, 4) + z(9, 3, 12) = (0, 0, 0)$$

$$x + 3y + 9z = 0 ; 4x + y + 3z = 0 ; 9x + 4y + 12z = 0$$

Consider $AX = 0$. Where

$$A = \begin{bmatrix} 1 & 3 & 9 \\ 4 & 1 & 3 \\ 9 & 4 & 12 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 4R_1$$

$$R_3 \rightarrow R_3 - 9R_1$$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & -11 & -33 \\ 0 & -23 & -69 \end{bmatrix}$$

$$R_2 \rightarrow -\frac{R_2}{11}$$

$$R_3 \rightarrow -\frac{R_3}{23}$$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank $[A] = 2 < 3$ (Number of unknowns)

So one unknown is constant and hence not always zero.

So the vectors are linearly dependent.

[OR]

$$v_3 = 3v_2$$

Vectors v_1, v_2, v_3 are linearly dependent.

11) If u, v, w are linearly independent vectors in $V(F)$, where F is the field of complex numbers, then $\{u+v, v+w, w+u\}$ is a linearly independent set of vectors.

Soln:- Let $a(u+v) + b(v+w) + c(w+u) = 0$, where $a, b, c \in F$

$$(a+c)u + (a+b)v + (b+c)w = 0 \rightarrow (1)$$

Since u, v, w are linearly independent.

$$\therefore \textcircled{1} \Rightarrow \begin{aligned} a+c &= 0 \\ a+b &= 0 \\ b+c &= 0. \end{aligned}$$

$$\therefore a=0, b=0, c=0.$$

$\therefore \{u+v, v+w, w+u\}$ is a linearly independent set of vectors.

12) Let V be the vector space of all 2×3 matrices over $\mathbb{R}$. S.T the matrices $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$ form a linearly independent set.

Solⁿ: - Let $aA + bB + cC = 0$ where $a, b, c \in \mathbb{R}$.

$$\Rightarrow a \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix} + b \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix} + c \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a+b+4c & a+b-c & -a-3b+2c \\ 3a-2b+c & -2a+0b-2c & 4a+5b+3c \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By equating the corresponding elements

$$\left. \begin{array}{l} 2a+b+4c=0 \\ a+b-c=0 \\ -a-3b+2c=0 \end{array} \right\} \rightarrow \textcircled{1} \quad \left. \begin{array}{l} 3a-2b+c=0 \\ -2a-2c=0 \\ 4a+5b+3c=0 \end{array} \right\} \rightarrow \textcircled{2}$$

On solving the system $\textcircled{1}$ of equations, we observe that the only solution is $a=0, b=0, c=0$. Clearly this solution also satisfies the system $\textcircled{2}$ of equations.

Hence the given set of matrices is linearly independent.

13) Show that the vectors $(1, 1, 2, 4), (2, -1, -5, 2), (1, -1, -4, 0)$ and $(2, 1, 1, 6)$ are linearly dependent.

14) Determine whether or not each of the following forms a basis, $x_1 = (1, 2, 3), x_2 = (3, -5, 6)$ in $\mathbb{R}^3$.

Solⁿ: - Basis in $\mathbb{R}^3$ must contain exactly 3 elements therefore x_1, x_2 does not form a basis in $\mathbb{R}^3$.

15) Determine whether or not each of the following form a basis $x_1 = (1, 2, 9)$, $x_2 = (2, -3, 4)$, $x_3 = (1, 3, 7)$, $x_4 = (2, 4, 8)$ in R^3 .

Soln:- Basis in R^3 must contain exactly 3 elements.
 $\therefore x_1, x_2, x_3, x_4$ does not form a basis in R^3 .

16) Determine whether or not each of the following forms a basis, $x_1 = (2, 2, 1)$, $x_2 = (1, 3, 7)$, $x_3 = (1, 2, 2)$ in R^3 .

Soln:- Three vectors in R^3 form a basis iff they are linearly independent.

$$x(2, 2, 1) + y(1, 3, 7) + z(1, 2, 2) = (0, 0, 0)$$

$$2x + y + z = 0; \quad 2x + 3y + 2z = 0; \quad x + 7y + 2z = 0.$$

Let $AX = 0$ where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 7 & 2 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 2 & 3 & 2 \\ 2 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 2R_1 \end{array}$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & -11 & -2 \\ 0 & -13 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow -R_2 \\ R_3 \rightarrow -R_3 \end{array}$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 13 & 3 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$R_3 \rightarrow \frac{R_3}{2}$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 11 & 2 \\ 0 & 1 & \frac{1}{2} \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_2} \begin{bmatrix} 1 & 7 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 11 & 2 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 11R_2}$$

$$A = \begin{bmatrix} 1 & 7 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{7}{2} \end{bmatrix}$$

Rank $[A] = 3 =$ Number of unknowns. So unique solution and solution is

$$x=0, y=0, z=0$$

Hence vectors x_1, x_2, x_3 are linearly independent and hence form a basis.

17) Let W be the subspace of $\mathbb{R}^5$ spanned by

$$x_1 = (1, 2, -1, 3, 4), x_2 = (2, 4, -2, 6, 8), x_3 = (1, 3, 2, 2, 6)$$

$$x_4 = (1, 4, 5, 1, 8), x_5 = (2, 7, 3, 3, 9)$$

Find a subset of vectors which forms a basis of W .

Solⁿ: - $A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{bmatrix}$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$R_4 \rightarrow R_4 - R_1$$

$$R_5 \rightarrow R_5 - 2R_1$$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{bmatrix}$$

$$R_4 \rightarrow R_4 - 2R_3, R_5 \rightarrow R_5 - 3R_3$$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -5 \end{bmatrix}$$

Number of non zero rows = 3

$\therefore \dim W = 3$ and $\{x_1, x_3, x_5\}$ forms a basis in W .

18) V is a vector space of polynomials over $\mathbb{R}$. Find a basis and dimension of the subspace W of V , spanned by the polynomials.

$$x_1 = t^3 - 2t^2 + 4t + 1, \quad x_2 = 2t^3 - 3t^2 + 9t - 1$$

$$x_3 = t^3 + 6t - 5, \quad x_4 = 2t^3 - 5t^2 + 7t + 5.$$

Soln: - $A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 2 & -3 & 9 & -1 \\ 1 & 0 & 6 & -5 \\ 2 & -5 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 2 & -6 \\ 0 & -1 & -1 & 3 \end{bmatrix}$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$R_4 \rightarrow R_4 - 2R_1$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$R_4 \rightarrow R_4 + R_2$$

$$A = \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Non Zero rows of Echelon matrix forms a basis.

Number of non zero rows = 2

$\therefore \dim W = 2$ and $\{x_1, x_2\}$ forms a basis in W .

Inner Product Space :-

Let V be a real vector space. Suppose to each pair of vectors $u, v \in V$ there is assigned a real number, denoted by $\langle u, v \rangle$. This function is called a (real) inner product on V if it satisfies the following axioms:

- (i) Linear Property: $\langle au_1 + bu_2, v \rangle = a\langle u_1, v \rangle + b\langle u_2, v \rangle$
- (ii) Symmetric Property: $\langle u, v \rangle = \langle v, u \rangle$
- (iii) Positive Definite Property: $\langle u, u \rangle \geq 0$; and $\langle u, u \rangle = 0$ if and only if $u = 0$.

Norm of a vector :

An inner product, $\langle u, u \rangle$ is nonnegative for any vector u .

$$\|u\| = \sqrt{\langle u, u \rangle} \quad \text{or} \quad \|u\|^2 = \langle u, u \rangle$$

This non negative number is called the norm or length of u .

* Every non zero vector v in V can be multiplied by the reciprocal of its length to obtain the unit vector $\hat{v} = \frac{1}{\|v\|} v$. which is a positive multiple of v .

This process is called normalizing v .

Orthogonality:-

Let V be an inner product space. The vectors $u, v \in V$ are said to be orthogonal and u is said to be orthogonal to v if

$$\langle u, v \rangle = 0.$$

Problems:-

1) Consider vectors $u = (1, 2, 4)$, $v = (2, -3, 5)$, $w = (4, 2, -3)$ in $\mathbb{R}^3$. Find a) $\langle u, v \rangle$ b) $\langle u, w \rangle$ c) $\langle v, w \rangle$ d) $\langle u+v, w \rangle$
e) $\|u\|$ f) $\|v\|$

a) $\langle u, v \rangle = 2 - 6 + 20 = 16$

b) $\langle u, w \rangle = 4 + 4 - 12 = -4$

c) $\langle v, w \rangle = 8 - 6 - 15 = -13$

d) $\langle u+v, w \rangle = (3, -1, 9) \cdot (4, 2, -3) = 12 - 2 - 27 = -17$

e) $\|u\| = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{1 + 4 + 16} = \sqrt{21}$

f) $\|v\| = \sqrt{4 + 9 + 25} = \sqrt{38}$

2) Consider the vectors $u = (1, 5)$ and $v = (3, 4)$ in $\mathbb{R}^2$. Find : a) $\langle u, v \rangle$ with respect to the usual inner product in $\mathbb{R}^2$.

b) $\|v\|$ using the inner product in $\mathbb{R}^2$.

3) Consider the following polynomials in $P(t)$ and inner product:

$$f(t) = t+2, \quad g(t) = 3t-2, \quad h(t) = t^2-2t-3 \text{ and}$$

$$\langle f, g \rangle = \int_0^1 f(t)g(t) dt.$$

(a) find $\langle f, g \rangle$ and $\langle f, h \rangle$ (b) find $\|f\|$ and $\|g\|$

(c) normalize f and g .

Solⁿ: (a) $\langle f, g \rangle = \int_0^1 (t+2)(3t-2) dt = \int_0^1 (3t^2 + 4t - 4) dt$

$$\langle f, g \rangle = \left[t^3 + 2t^2 - 4t \right]_0^1 = -1.$$

$$\langle f, h \rangle = \int_0^1 (t+2)(t^2-2t-3) dt = \left[\frac{t^4}{4} - \frac{7t^3}{2} + 6t \right]_0^1 = -\frac{37}{4}$$

(b) $\langle f, f \rangle = \int_0^1 (t+2)(t+2) dt = \frac{19}{3}$; $\|f\| = \frac{\sqrt{19}}{\sqrt{3}} = \frac{\sqrt{57}}{3}$.

$$\langle g, g \rangle = \int_0^1 (3t-2)(3t-2) dt = 1$$
 ; $\|g\| = \sqrt{1} = 1$

(c) Since $\|f\| = \frac{\sqrt{57}}{3}$ and g is already a unit vector,

$$\hat{f} = \frac{1}{\|f\|} f = \frac{3}{\sqrt{57}} (t+2)$$

$$\hat{g} = \frac{1}{\|g\|} g = 3t-2.$$

4) Let $M = M_{2,3}$ with inner product. $\langle A, B \rangle = \text{tr} \langle B^T A \rangle$

and let $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $C = \begin{bmatrix} 3 & -5 & 2 \\ 1 & 0 & -4 \end{bmatrix}$

Find (a) $\langle A, B \rangle$, $\langle A, C \rangle$, $\langle B, C \rangle$

(b) $\langle 2A+3B, 4C \rangle$ (c) $\|A\|$ and $\|B\|$

$$(a) \quad \langle A, B \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij} b_{ij}$$

$$\langle A, B \rangle = 9 + 16 + 21 + 24 + 25 + 24 = 119$$

$$\langle A, C \rangle = 27 - 40 + 14 + 6 + 0 - 16 = -9$$

$$\langle B, C \rangle = 3 - 10 + 6 + 4 + 0 - 24 = -21$$

$$(b) \quad 2A + 3B = \begin{bmatrix} 21 & 22 & 23 \\ 24 & 25 & 26 \end{bmatrix} \quad 4C = \begin{bmatrix} 12 & -20 & 8 \\ 4 & 0 & -16 \end{bmatrix}$$

$$\langle 2A + 3B, 4C \rangle = 252 - 440 + 96 + 0 - 416 = -324$$

$$(c) \quad \|A\|^2 = \langle A, A \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij}^2, \text{ the sum of the squares of all the elements of } A.$$

$$\|A\|^2 = \langle A, A \rangle = 9^2 + 8^2 + 7^2 + 6^2 + 5^2 + 4^2 = 271 \Rightarrow \|A\| = \sqrt{271}$$

$$\|B\|^2 = \langle B, B \rangle = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91 \Rightarrow \|B\| = \sqrt{91}$$

4) Verify the vectors $u = (1, 1, 1)$, $v = (1, 2, -3)$ & $w = (1, -4, 3)$ in $\mathbb{R}^3$ are orthogonal or not.

Soln:- $\langle u, v \rangle = 1 + 2 - 3 = 0$, $\langle u, w \rangle = 1 - 4 + 3 = 0$, $\langle v, w \rangle = 1 - 8 - 9 = -16$

Thus u is orthogonal to v and w ,
 v & w are not orthogonal.

VECTOR SPACE

ROW SPACE & COLUMN SPACE:

Let $A = [P_{ij}]$ be an $m \times n$ real matrix that maps the vectors in R^n to vectors in R^m . The four fundamental subspaces be associated with A , two in R^n and two in R^m , are:

1. Row space of A , the set of all x for which Ax is non zero.
2. Null space of A , the set of all x for which $Ax = 0$.
3. Column space of A , the set of all y for which $A^T y$ is nonzero.
4. Left null space of A , the set of all y for which $A^T y = 0$.

The row space and null space are orthogonal to each other and span all of R^n . The column space and the left null space are also orthogonal to each other and span all of R^m .

ROW SPACE, COLUMN SPACE AND NULLSPACE

Let A be a $m \times n$ matrix, then

- (i) The **row space** of A is the subspace of R^n spanned by the row vectors of A
- (ii) The **column space** of A is the subspace of R^m spanned by the column vectors of A .
- (iii) The solution space of the homogeneous system of equations $Ax = 0$, which is a subspace of R^n , is called the **null space** (or kernel) of A .

EXAMPLE - 1 :

1. Find bases for the row space, the column space of the matrix A

Step 1 : Reduce the matrix to row echelon form

Step 2: Identify the row space

Step 3 : Identify the column space

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

Step-1 Reduce the matrix to row echelon form:

$$A \sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ -2 & -5 & 8 & 0 & -17 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow 2R_1 + R_2 \\ R_3 \rightarrow -3R_1 + R_3 \\ R_4 \rightarrow -R_1 + R_4 \end{array}$$

$$A \sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 2 & -4 & 4 & -14 \\ 0 & 4 & -8 & 4 & -8 \end{bmatrix} \quad \begin{array}{l} R_3 \rightarrow -2R_2 + R_3 \\ R_4 \rightarrow -4R_2 + R_4 \end{array}$$

$$A \sim \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 20 \end{bmatrix} \quad R_3 \leftrightarrow R_4$$

$$A \sim \begin{array}{ccccc} & C_1 & C_2 & C_3 & C_4 & C_5 \\ \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & R_1 & R_2 & R_3 & R_4 \\ & & & R_3 \rightarrow R_3 \div -4 & \end{array}$$

$$\begin{array}{c}
 \begin{array}{ccccc}
 & C_1 & C_2 & & C_3 \\
 A \sim \left[\begin{array}{ccccc}
 \textcircled{1} & 3 & -5 & 0 & 10 \\
 0 & \textcircled{1} & -2 & 2 & 7 \\
 0 & 0 & 0 & \textcircled{1} & -5 \\
 0 & 0 & 0 & 0 & 0
 \end{array} \right] & \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4
 \end{array}
 \end{array}
 \end{array}$$

Step 2: Row Space: Non Zero rows are R_1, R_2, R_3 . Row space of the matrix is the span of these vectors.

$$\text{Row}(A) = \text{Span}\{ (1, 3, -5, 0, 10), (0, 1, -2, 2, 7), (0, 0, 0, 1, -5) \}$$

Step 3: Column Space: For the column space, the pivots are in C_1, C_2 and C_4 (First non zero entry in a row). Hence **column 1, 2 & 4 from original matrix form a basis.**

$$\text{Basis for Col } A : \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

EXAMPLE 2: Find bases for the row space, the column space of the matrix A

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 & -1 \\ 2 & 4 & -2 & -3 & -1 \\ 5 & 10 & -5 & -3 & -1 \\ -3 & -6 & 3 & 2 & 1 \end{bmatrix}$$

We first reduce A to row-echelon form:

$$\begin{aligned}
 A &\stackrel{1}{\sim} \begin{bmatrix} 1 & 2 & -1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & -4 & -2 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & 2 & -1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \\
 &\stackrel{3}{\sim} \begin{bmatrix} 1 & 2 & -1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \stackrel{4}{\sim} \begin{bmatrix} 1 & 2 & -1 & -2 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

Step 2: Row Space: Non Zero rows are R1, R2, R3. Row space of the matrix is the span of these vectors.

$$\text{Row}(A) = \text{Span}\{ (1,2,-1,-2,-1), (0,0,0,1,1), (0,0,0,0,1) \}$$

Step 3: Column Space: For the column space, the pivots are in C1,C4 and C5 (First non zero entry in a row). Hence **column 1,2 & 4 from original matrix form a basis.**

$$\text{Basis for Col A : } \left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$



Practice Problems:

Find bases for the row space, the column space of the following matrix:

$$1. A = \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix}.$$

$$2. A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ 3 & 4 & -11 & 7 \end{bmatrix}.$$

$$3. A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix}.$$

$$4. A = \begin{bmatrix} 0 & 3 & 1 \\ 0 & -6 & -2 \\ 0 & 12 & 4 \end{bmatrix}.$$

$$5. A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 6 & -3 & 5 \\ 1 & 2 & -1 & -1 \\ 5 & 10 & -5 & 7 \end{bmatrix}.$$

$$6. A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 1 & 1 & -2 & 6 \\ 3 & 1 & 4 & 2 \end{bmatrix}.$$

Module-5:

Linear Transformations

* Linear Transformation:

Let V and W be two vector spaces over the same field F . The mapping $T: V \rightarrow W$ is said to be a linear transformation, if

$$(i) T(x+y) = T(x) + T(y)$$

$$(ii) T(c \cdot x) = c \cdot T(x).$$

where $x, y \in V$ & $c \in F$.

(i) Show that the transformation $T: V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ defined by $T(x, y) = (x+y, x-y)$ is a linear transformation

Soln: Let $u = (x_1, y_1)$ & $v = (x_2, y_2) \in V_2(\mathbb{R})$

$$T(u) = T(x_1, y_1) = (x_1 + y_1, x_1 - y_1)$$

$$T(v) = T(x_2, y_2) = (x_2 + y_2, x_2 - y_2)$$

$$\begin{aligned} \text{Consider } T(u+v) &= T(x_1+x_2, y_1+y_2) \\ &= [x_1+x_2+y_1+y_2, x_1+x_2-(y_1+y_2)] \\ &= [(x_1+y_1), x_1-y_1] + [(x_2+y_2), x_2-y_2] \\ &= T(u) + T(v) \end{aligned}$$

$$\text{Hence } T(u+v) = T(u) + T(v)$$

Next, consider $T(c \cdot u)$ where $c \in F$

$$\begin{aligned} T(c \cdot u) &= T(cx_1, cy_1) = (cx_1 + cy_1, cx_1 - cy_1) \\ &= c(x_1 + y_1, x_1 - y_1) \\ &= c \cdot T(u) \end{aligned}$$

Thus T is the linear transformation.

2) If S and T are two transformations from $V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ defined by $S(x, y) = (3x + 2y, 3x - 4y)$ and $T(x, y) = (x + 3, y - 2)$. S is linear but T is not linear.

Soln: Consider $S(x, y) = (3x + 2y, 3x - 4y)$

Let $u = (x_1, y_1)$ & $v = (x_2, y_2) \in V_2(\mathbb{R})$

$$S(u) = S(x_1, y_1) = (3x_1 + 2y_1, 3x_1 - 4y_1)$$

$$S(v) = S(x_2, y_2) = (3x_2 + 2y_2, 3x_2 - 4y_2)$$

$$S(u+v) = S(x_1+x_2, y_1+y_2)$$

$$= [3(x_1+x_2) + 2(y_1+y_2), 3(x_1+x_2) - 4(y_1+y_2)]$$

$$= [(3x_1 + 2y_1) + (3x_2 + 2y_2), (3x_1 - 4y_1) + (3x_2 - 4y_2)]$$

$$= [(3x_1 + 2y_1, 3x_1 - 4y_1) + (3x_2 + 2y_2, 3x_2 - 4y_2)]$$

$$\text{Hence } S(u+v) = S(u) + S(v)$$

Next consider $S(c \cdot u)$, where $c \in F$

$$S(c \cdot u) = S(c \cdot (x_1, y_1)) = S(cx_1, cy_1)$$

$$= (3cx_1 + 2cy_1, 3cx_1 - 4cy_1)$$

$$= c(3x_1 + 2y_1, 3x_1 - 4y_1)$$

$$S(c \cdot u) = c \cdot S(u)$$

Hence the transformation S is linear.

* Now consider $T(x, y) = (x + 3, y - 2)$

$$T(u) = T(x_1, y_1) = (x_1 + 3, y_1 - 2)$$

$$T(v) = T(x_2, y_2) = (x_2 + 3, y_2 - 2)$$

$$T(u+v) = T(x_1+x_2, y_1+y_2) = (x_1+x_2+3, y_1+y_2-2)$$

$$T(u) + T(v) = (x_1+3, y_1-2) + (x_2+3, y_2-2) = (x_1+x_2+6, y_1+y_2-4)$$

$$T(u) + T(v) \neq T(u+v)$$

Thus, the T is not a linear transformation.

③ Show that the transformation $T: V_3(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ defined by $T(x, y, z) = (y-x, y-z)$ is a linear transformation.

Soln: $\rightarrow$ Let $u = (x_1, y_1, z_1)$ & $v = (x_2, y_2, z_2) \in V_3(\mathbb{R})$

$$T(u) = T(x_1, y_1, z_1) = (y_1 - x_1, y_1 - z_1)$$

$$T(v) = T(x_2, y_2, z_2) = (y_2 - x_2, y_2 - z_2)$$

$$\text{Now } T(u+v) = T(x_1+x_2, y_1+y_2, z_1+z_2)$$

$$= [(y_1+y_2) - (x_1+x_2), (y_1+y_2) - (z_1+z_2)]$$

$$= [(y_1 - x_1) + (y_2 - x_2), (y_1 - z_1) + (y_2 - z_2)]$$

$$= [(y_1 - x_1, y_1 - z_1) + (y_2 - x_2, y_2 - z_2)]$$

$$T(u+v) = T(u) + T(v)$$

Next consider, $T(c \cdot u)$ where $c \in \mathbb{F}$

$$T(c \cdot u) = T(c \cdot (x_1, y_1, z_1)) = T(cx_1, cy_1, cz_1)$$

$$= (cy_1 - cx_1, cy_1 - cz_1) = c(y_1 - x_1, y_1 - z_1)$$

$$T(c \cdot u) = c \cdot T(u)$$

Hence T is a linear transformation.

4) If T is a transformation from $V_3(\mathbb{R})$ to $V_2(\mathbb{R})$ defined by $T(a, b) = (a \cos \theta - b \sin \theta, a \sin \theta + b \cos \theta)$, decide whether the T is linear or not

Soln: Let $u = (a_1, b_1)$ & $v = (a_2, b_2) \in V_2(\mathbb{R})$

$$T(u) = T(a_1, b_1) = (a_1 \cos \theta - b_1 \sin \theta, a_1 \sin \theta + b_1 \cos \theta)$$

$$T(v) = T(a_2, b_2) = (a_2 \cos \theta - b_2 \sin \theta, a_2 \sin \theta + b_2 \cos \theta)$$

Consider

$$T(u+v) = T(a_1+a_2, b_1+b_2)$$

$$= [(a_1+a_2) \cos \theta - (b_1+b_2) \sin \theta, (a_1+a_2) \sin \theta + (b_1+b_2) \cos \theta]$$

$$= [(a_1 \cos \theta - b_1 \sin \theta) + (a_2 \cos \theta - b_2 \sin \theta), (a_1 \sin \theta + b_1 \cos \theta) + (a_2 \sin \theta + b_2 \cos \theta)]$$

$$= [(a_1 \cos \theta - b_1 \sin \theta, a_1 \sin \theta + b_1 \cos \theta) + (a_2 \cos \theta - b_2 \sin \theta, a_2 \sin \theta + b_2 \cos \theta)]$$

$$T(u+v) = T(u) + T(v)$$

Next consider $T(c \cdot u) = T(c \cdot (a_1, b_1)) = T(ca_1, cb_1)$

$$T(c \cdot u) = (ca_1 \cos \theta - cb_1 \sin \theta, ca_1 \sin \theta + cb_1 \cos \theta) \\ = c \cdot (a_1 \cos \theta - b_1 \sin \theta, a_1 \sin \theta + b_1 \cos \theta)$$

$$T(c \cdot u) = c \cdot T(u)$$

Hence we conclude that T is a linear transformation.

5) Find the linear map $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$f(1, 2) = (3, 0) \text{ \& \& } f(2, 1) = (1, 2)$$

Soln: The elements $(1, 2)$ & $(2, 1) \in \mathbb{R}^2$ form a basis
& let $(x, y) \in \mathbb{R}^2$

$$\therefore (x, y) = c_1(1, 2) + c_2(2, 1) \text{ --- (a)}$$

$$\Rightarrow (x, y) = (c_1 + 2c_2, 2c_1 + c_2)$$

$$\rightarrow c_1 + 2c_2 = x \quad ; \quad 2c_1 + c_2 = y$$

Solve (1) & (2):

$$2 \times (1) - (2) \Rightarrow \begin{array}{r} 2c_1 + 4c_2 = 2x \\ -(2c_1 + c_2 = y) \\ \hline 3c_2 = 2x - y \end{array}$$

$$c_2 = \frac{1}{3}(2x - y)$$

from (1)

$$c_1 = x - 2c_2$$

$$c_1 = x - \frac{2}{3}(2x - y)$$

$$= x - \frac{4x}{3} + \frac{2}{3}y$$

$$= -\frac{x}{3} + \frac{2}{3}y$$

$$c_1 = \frac{1}{3}(2y - x)$$

Sub c_1, c_2 in (a)

$$\therefore (x, y) = \frac{1}{3}(2y - x)(1, 2) + \frac{1}{3}(2x - y)(2, 1)$$

$$f(x, y) = \left[\frac{1}{3}(2y - x)f(1, 2) + \frac{1}{3}(2x - y)f(2, 1) \right]$$

$$= \left[\frac{1}{3}(2y - x)(3, 0) + \frac{1}{3}(2x - y)(1, 2) \right]$$

$$= \left[(2y - x) + \frac{1}{3}(2x - y), 0 + \frac{2}{3}(2x - y) \right]$$

$$= \left[\frac{2x}{3} - x + 2y - \frac{y}{3}, \frac{4x}{3} - \frac{2y}{3} \right]$$

$$f(x, y) = \left[\frac{2x - 3x + 6y - y}{3}, \frac{2(2x - y)}{3} \right] = \left[\frac{1}{3}(5y - x), \frac{2}{3}(2x - y) \right]$$

$f(x, y)$ is the required linear transformation

6) Determine the linear transformation $T: V_3(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ such that $T(e_1) = (-1, 0)$; $T(e_2) = (1, 1)$; $T(e_3) = (0, -1)$ where e_1, e_2, e_3 is the standard basis of $V_3(\mathbb{R})$

$\Rightarrow$ Here $e_1 = (1, 0, 0)$; $e_2 = (0, 1, 0)$; $e_3 = (0, 0, 1)$ are the standard basis of $V_3(\mathbb{R})$

Let $(x, y, z) \in \mathbb{R}^3$ be arbitrary

$$(x, y, z) = c_1(1, 0, 0) + c_2(0, 1, 0) + c_3(0, 0, 1)$$

$$(x, y, z) = (c_1, c_2, c_3)$$

$$\Rightarrow c_1 = x, c_2 = y, c_3 = z$$

$$(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$$

$$\Rightarrow T(x, y, z) = xT(1, 0, 0) + yT(0, 1, 0) + zT(0, 0, 1)$$
$$= xT(e_1) + yT(e_2) + zT(e_3)$$

$$T(x, y, z) = x(-1, 0) + y(1, 1) + z(0, -1)$$
$$= (-x + y + 0, 0 + y - z)$$

Thus the linear transformation is

$$T(x, y, z) = (y - x, y - z)$$
$$\underline{\underline{\quad}}$$

* Algebra of Linear Transformation:

Let $T_1: V \rightarrow W$ and $T_2: V \rightarrow W$ be any two linear transformations. Then the sum of T_1 and T_2 denoted by $T_1 + T_2$ is defined as

$$(T_1 + T_2)(x) = T_1(x) + T_2(x), \forall x \in V.$$

Also if 'c' is any scalar then the scalar multiple of T denoted by cT is defined as.

$$(cT)(x) = cT(x), \forall x \in V$$

Further the composition of T_1 and T_2 denoted by $T_1 \circ T_2$ is defined by $T_1 \circ T_2(x) = T_1[T_2(x)], \forall x \in V$

It may be noted that $T_1 \circ T_2$ is denoted by $T_1 T_2$ and

$T \circ T$ is denoted by T^2 .

① Let T_1 & T_2 be two linear transformation from $V_3(\mathbb{R})$ to $V_2(\mathbb{R})$ defined by $T_1(x, y, z) = (x+y, y+z)$ and $T_2(x, y, z) = (y+z, z+x)$ find $T_1 + T_2$, $2T_1 + 3T_2$

$$\begin{aligned} \Rightarrow (T_1 + T_2)(x, y, z) &= T_1(x, y, z) + T_2(x, y, z) \\ &= (x+y, y+z) + (y+z, z+x) \\ &= (x+y+y+z, y+z+z+x) \\ T_1 + T_2 &= (x+2y+z, y+2z+x) \end{aligned}$$

$$\text{Also, } \underline{(2T_1 + 3T_2)(x, y, z)} = 2T_1(x, y, z) + 3T_2(x, y, z)$$

$$\begin{aligned} &= 2(x+y, y+z) + 3(y+z, z+x) \\ &= (2x+2y, 2y+2z) + (3y+3z, 3z+3x) \\ &= (2x+2y+3y+3z, 2y+2z+3z+3x) \end{aligned}$$

$$\begin{aligned} 2T_1 + 3T_2 &= (2x+5y+3z, 3x+5z+2y) \\ &= (2x+5y+3z, 3x+2y+5z) \end{aligned}$$

2) If T_1 and T_2 are two linear transformations from $V_3(\mathbb{R})$ to $V_2(\mathbb{R})$ defined by
 $T_1(x, y, z) = (2x+y, y+2z)$ and $T_2(x, y, z) = (y+2z, 2z+x)$
 determine $T_1 + T_2$ and $3T_1 - 2T_2$

Soln: $(T_1 + T_2)(x, y, z) = T_1(x, y, z) + T_2(x, y, z)$
 $= (2x+y, y+2z) + (y+2z, 2z+x)$
 $= (2x+y+y+2z, y+2z+2z+x)$
 $T_1 + T_2 = (2x+2y+2z, x+y+4z)$

Next, $(3T_1 - 2T_2)(x, y, z) = 3T_1(x, y, z) - 2T_2(x, y, z)$
 $= 3(2x+y, y+2z) - 2(y+2z, 2z+x)$
 $= (6x+3y, 3y+6z) - (2y+4z, 4z+2x)$
 $= (6x+3y-2y-4z, 3y+6z-4z-2x)$
 $3T_1 - 2T_2 = (6x+y-4z, -2x+3y+2z)$

③ If T_1 and T_2 are linear operators from $V_2(\mathbb{R})$ to $V_2(\mathbb{R})$ defined by $T_1(x, y) = (x+y, x-y)$, $T_2(x, y) = (x-y, x+y)$
 Compute $T_1 T_2$, $T_2 T_1$, T_1^2 , T_2^2

Soln: $T_1 T_2 = T_1 \circ T_2(x, y) = T_1[T_2(x, y)]$
 $= T_1(x-y, x+y)$
 $= (x-y+x+y, x-y-x-y)$

$T_1 T_2 = (2x, -2y)$

$$\begin{aligned}
 * \quad T_2 T_1 &= T_2 \circ T_1(x, y) = T_2[T_1(x, y)] \\
 &= T_2(x+y, x-y) \\
 &= (x+y-x+y, x+y+x-y) \\
 \underline{\underline{T_2 T_1}} &= (2y, 2x)
 \end{aligned}$$

$$\begin{aligned}
 * \quad T_1^2 &= T_1 \circ T_1 = T_1[T_1(x, y)] \\
 &= T_1(x+y, x-y) \\
 &= (x+y+x-y, x+y-x+y) \\
 \underline{\underline{T_1^2}} &= (2x, 2y)
 \end{aligned}$$

$$\begin{aligned}
 * \quad T_2^2 &= T_2 \circ T_2 = T_2 \circ T_2(x, y) = T_2[T_2(x, y)] \\
 &= T_2(x-y, x+y) \\
 &= (x-y-x-y, x-y+x+y) \\
 \underline{\underline{T_2^2}} &= (-2y, 2x)
 \end{aligned}$$

* Matrix of a Linear Transformation

① Let $T: V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ be defined as $T(x, y) = (-x, -y)$ w.r.t. the standard basis. Find the matrix of T .

Soln: The standard basis of $V_2(\mathbb{R})$ are

$$e_1 = (1, 0) \text{ \& } e_2 = (0, 1)$$

$$T(e_1) = T(1, 0) = (-1, 0)$$

$$T(e_2) = T(0, 1) = (0, -1)$$

* The co-efficient matrix $M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

* The matrix of the linear transformation is $M_T = M'$

$$\underline{M_T = M(T) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}$$

2) Let $T: V_3(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ be defined as $T(x, y, z) = (x+y, y+z)$. Find the matrix of T w.r.t. the basis $\{(1, 1, 1), (1, 0, 0), (1, 1, 0)\}$ and also w.r.t. the standard basis.

$\Rightarrow$ (i) Given basis $\{(1, 1, 1), (1, 0, 0), (1, 1, 0)\}$

$$T(x, y, z) = (x+y, y+z)$$

$$\therefore T(1, 1, 1) = (2, 2) ; T(1, 0, 0) = (1, 0) ; T(1, 1, 0) = (2, 1)$$

* The co-efficient matrix $M = \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$

* The matrix of L.T is $\underline{M_T = M' = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}}$

case (ii): standard basis of $\mathbb{R}^3$ are $\{(\underbrace{1, 0, 0}_{e_1}), (\underbrace{0, 1, 0}_{e_2}), (\underbrace{0, 0, 1}_{e_3})\}$

$$T(x, y, z) = (x+y, y+z)$$

$$\therefore T(e_1) = T(1, 0, 0) = (1, 0)$$

$$T(e_2) = T(0, 1, 0) = (1, 1)$$

$$T(e_3) = T(0, 0, 1) = (0, 1)$$

* The co-efficient matrix $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$

* The matrix of T is $M_T = M(T) = M' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

③ Find the matrix of the linear transformation $T: V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ defined by $T(x, y) = (x, -y)$ relative to the basis; $X = \{e_1, e_2\}$ and $Y = \{(1, 1), (1, -1)\}$

$$\Rightarrow e_1 = (1, 0) \text{ \& } e_2 = (0, 1)$$

$$T(x, y) = (x, -y)$$

$$T(e_1) = T(1, 0) = (1, 0)$$

$$T(e_2) = T(0, 1) = (0, -1)$$

Now we need to write $(1, 0)$ & $(0, -1)$ as linear combination of elements in Y .

$$\rightarrow (1, 0) = c_1(1, 1) + c_2(1, -1) \quad \& \quad (0, -1) = c_3(1, 1) + c_4(1, -1)$$

$$(1, 0) = (c_1 + c_2, c_1 - c_2) \quad ; \quad (0, -1) = (c_3 + c_4, c_3 - c_4)$$

$$c_1 + c_2 = 1 \quad ; \quad c_1 - c_2 = 0$$

— ①

— ②

Solve ① & ②

$$c_1 + c_2 = 1$$

$$c_1 - c_2 = 0$$

$$\underline{2c_1 = 1}$$

$$\boxed{c_1 = 1/2}$$

; from ②

$$c_1 = c_2$$

$$\boxed{c_2 = 1/2}$$

$$c_3 + c_4 = 0, \quad c_3 - c_4 = -1$$

— ③

— ④

Solve ③ & ④

$$c_3 + c_4 = 0$$

$$c_3 - c_4 = -1$$

$$\underline{2c_3 = -1}$$

$$\boxed{c_3 = -1/2}$$

from ③

$$c_4 = -c_3$$

$$c_4 = -(-1/2)$$

$$\boxed{c_4 = 1/2}$$

Hence we have

$$T(1,0) = (1,0) = \frac{1}{2}(1,1) + \frac{1}{2}(1,-1) \text{ \& } T(0,1) = (0,-1) = -\frac{1}{2}(1,1) + \frac{1}{2}(1,-1)$$

$$\therefore \text{The coefficient matrix } M = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\therefore \text{The matrix of } T \text{ is } M_T = M' = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(4) Find Matrix of the linear transformation $T: V_2(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ defined by $T(x,y) = (x, -y)$ relative to the basis $X = \{(1,1), (1,0)\}$ and $Y = \{(2,3), (4,5)\}$

$$\Rightarrow T(1,1) = (1,-1) ; T(1,0) = (1,0)$$

Now we need to write $(1,-1)$ & $(1,0)$ as linear combination of elements in Y

$$(1,-1) = c_1(2,3) + c_2(4,5) \text{ \& } (1,0) = c_3(2,3) + c_4(4,5)$$

$$(1,-1) = (2c_1 + 4c_2, 3c_1 + 5c_2) \text{ \& } (1,0) = (2c_3 + 4c_4, 3c_3 + 5c_4)$$

$$2c_1 + 4c_2 = 1 ; 3c_1 + 5c_2 = -1 \text{ \& } 2c_3 + 4c_4 = 1 ; 3c_3 + 5c_4 = 0 \quad \text{--- (4)}$$

Solve (1) & (2)

$$3 \times (1) - 2 \times (2) \Rightarrow$$

$$\Rightarrow 6c_1 + 12c_2 = 3$$

$$-6c_1 + 10c_2 = -2$$

$$2c_2 = 5$$

$$\boxed{c_2 = \frac{5}{2}}$$

$$\text{From (1)} \Rightarrow 2c_1 = 1 - 4c_2$$

$$2c_1 = 1 - 4 \times \frac{5}{2} = 1 - 10$$

$$\boxed{c_1 = -\frac{9}{2}}$$

Solve (3) & (4)

$$3 \times (3) - 2 \times (4) \Rightarrow$$

$$6c_3 + 12c_4 = 3$$

$$6c_3 + 10c_4 = 0$$

$$\boxed{c_4 = \frac{3}{2}}$$

$$\text{From (3)} \Rightarrow 2c_3 = 1 - 4c_4$$

$$2c_3 = 1 - 4 \times \frac{3}{2} = 1 - 6$$

$$\boxed{c_3 = -\frac{5}{2}}$$

∴ The co-efficient matrix $M = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$

$$M = \begin{bmatrix} -9/2 & 5/2 \\ -5/2 & 3/2 \end{bmatrix}$$

∴ The matrix of T , $M_T = M' = \begin{bmatrix} -9/2 & -5/2 \\ 5/2 & 3/2 \end{bmatrix}$

5) Determine the matrix of the linear transformation $T: V_2(R) \rightarrow V_3(R)$ defined by $T(x, y) = (-x+2y, y, 3x+2y)$ relative to the basis $X = \{(1, 1), (-1, 1)\}$ & $Y = \{(1, 1, 1), (1, -1, 1), (0, 0, 1)\}$

⇒ By the defⁿ of the given $T(x, y)$ we have

$$T(1, 1) = (1, 1, 6)$$

$$T(-1, 1) = (3, 1, 0)$$

$$\text{Let } (1, 1, 6) = c_1(1, 1, 1) + c_2(1, -1, 1) + c_3(0, 0, 1)$$

$$\& (3, 1, 0) = c_4(1, 1, 1) + c_5(1, -1, 1) + c_6(0, 0, 1)$$

$$\Rightarrow (1, 1, 6) = (c_1+c_2, c_1-c_2, c_1+c_2+c_3) \& (3, 1, 0) = (c_4+c_5, c_4-c_5, c_4+c_5+c_6)$$

$$c_1+c_2=1; \quad c_1-c_2=1; \quad c_1+c_2+c_3=6 \& c_4+c_5=3; \quad c_4-c_5=1; \quad c_4+c_5+c_6=0$$

- (a) - (b) - (c) - (1) - (2) - (3)

Solve (a) & (b)

$$c_1 + c_2 = 1$$

$$c_1 - c_2 = 1$$

$$2c_1 = 2$$

$$\boxed{c_1 = 1}$$

From (b) $\Rightarrow c_2 = 1 - c_1$

$$c_2 = 1 - 1 = 0$$

$$\boxed{c_2 = 0}$$

From (c) $\Rightarrow$

$$c_3 = 6 - c_1 - c_2$$

$$c_3 = 6 - 1 - 0$$

$$\boxed{c_3 = 5}$$

Solve (1) & (2)

$$c_4 + c_5 = 3$$

$$c_4 - c_5 = 1$$

$$2c_4 = 4$$

$$\boxed{c_4 = 2}$$

From (1) $\Rightarrow$

$$c_5 = 3 - c_4$$

$$c_5 = 3 - 2$$

$$\boxed{c_5 = 1}$$

From (3) $\Rightarrow$

$$c_6 = 0 - c_4 - c_5$$

$$c_6 = -2 - 1$$

$$\boxed{c_6 = -3}$$

$\therefore$ the coefficient matrix $M = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \end{bmatrix}$

$$M = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & -3 \end{bmatrix}$$

Thus, the required matrix of T $M_T = M' = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 5 & -3 \end{bmatrix}$

6) Find the linear transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfying $T(1, 1) = (0, 1)$ and $T(-1, 1) = (3, 2)$

Soln: Let $(x, y) \in \mathbb{R}^2$ be arbitrary

$$\therefore (x, y) = c_1(1, 1) + c_2(-1, 1) \quad \text{--- (*)}$$

$$\Rightarrow (x, y) = (c_1 - c_2, c_1 + c_2)$$

$$\begin{array}{l} c_1 - c_2 = x \quad \text{--- (1)} \\ c_1 + c_2 = y \quad \text{--- (2)} \end{array}$$

Solve (1) & (2)

$$c_1 - c_2 = x$$

$$c_1 + c_2 = y$$

$$\hline 2c_1 = x + y$$

$$\boxed{c_1 = \frac{x+y}{2}}$$

$$\text{from (1)} \Rightarrow c_1 - c_2 = x$$

$$c_2 = c_1 - x$$

$$= c_1 - x = \frac{x+y}{2} - x$$

$$\boxed{c_2 = \frac{x+y-2x}{2} = \frac{y-x}{2}}$$

Sub c_1 & c_2 in (*)

$$\therefore (x, y) = \frac{1}{2}(x+y)(1, 1) + \frac{(y-x)}{2}(-1, 1)$$

$$T(x, y) = \frac{1}{2}(x+y)T(1, 1) + \frac{1}{2}(y-x)T(-1, 1)$$

$$= \frac{1}{2}(x+y)[0, 1] + \frac{1}{2}(y-x)[3, 2]$$

$$= \left[0 + \frac{3}{2}(y-x), \frac{1}{2}(x+y) + \frac{2}{2}(y-x) \right]$$

$T(x, y) = \left[\frac{-3x}{2} + \frac{3y}{2}, \frac{-x}{2} + \frac{3y}{2} \right]$ is the require linear transformation.

7. Given $A = \begin{bmatrix} -1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$ and the basis

$$X = \{(1, 1, 1), (1, 2, 3), (1, 1, 0)\} \text{ \& } Y = \{(1, 0), (2, -1)\}$$

determine the linear transformation $T: V_3(R) \rightarrow V_2(R)$

Soln: $T(1, 1, 1) = -1(1, 0) + 3(2, -1) = (5, -3)$

$$T(1, 2, 3) = -1(1, 0) + 1(2, -1) = (1, -1)$$

$$T(1, 1, 0) = 2(1, 0) + 0(2, -1) = (2, 0)$$

Let $(x, y, z) \in V_3(R)$ be arbitrary.

$$\therefore (x, y, z) = c_1(1, 1, 1) + c_2(1, 2, 3) + c_3(1, 1, 0) \quad \text{--- (1)}$$

$$\Rightarrow (x, y, z) = (c_1 + c_2 + c_3, c_1 + 2c_2 + c_3, c_1 + 3c_2)$$

$$\Rightarrow c_1 + c_2 + c_3 = x \quad \text{--- (a)}; \quad c_1 + 2c_2 + c_3 = y \quad \text{--- (b)}; \quad c_1 + 3c_2 = z \quad \text{--- (c)}$$

Solve (a) & (b)

$$\begin{array}{r} c_1 + c_2 + c_3 = x \\ c_1 + 2c_2 + c_3 = y \\ \hline (-) \quad (-) \quad (-) \quad (-) \\ -c_2 = x - y \\ \hline \boxed{c_2 = y - x} \end{array}$$

From (c) $\Rightarrow c_1 = z - 3c_2$

$$c_1 = z - 3(y - x)$$

$$c_1 = z - 3y + 3x$$

$$\boxed{c_1 = 3x - 3y + z}$$

From (a) $\Rightarrow c_3 = x - c_1 - c_2$

$$c_3 = x - (3x - 3y + z) - (y - x) = x - 3x + 3y - z - y + x$$

$$\boxed{c_3 = -x + 2y - z}$$

Sub c_1, c_2, c_3 in (1)

$$\therefore (x, y, z) = (3x - 3y + z)(1, 1, 1) + (y - x)(1, 2, 3) + (-x + 2y - z)(1, 1, 0)$$

$$T(x, y, z) = (3x - 3y + z)T(1, 1, 1) + (y - x)T(1, 2, 3) + (-x + 2y - z)T(1, 1, 0)$$

$$= (3x - 3y + z)(5, -3) + (y - x)(1, -1) + (-x + 2y - z)(2, 0)$$

$$= (15x - 15y + 5z + y - x - 2x + 4y - 2z, -9x + 9y - 3z - y + x)$$

$$T(x, y, z) = (12x - 10y + 3z, -8x + 8y - 3z)$$

Singular and Non-Singular Linear Transformation

Let $F: V \rightarrow U$ be a linear mapping

F is said to be singular, if the image of some non-zero vector v is 0, that is if there exists $v \neq 0$, such that

$$F(v) = 0$$

$F: V \rightarrow U$ is said to be non-singular if the zero vector 0 is the only vector whose image under F is 0.

In other words $\text{Ker}(F) = \{0\}$

NOTE: $|A| = 0$, then A is singular matrix
 $|A| \neq 0$, then A is non-singular matrix.

Invertible Linear Transformation

Let $F: V \rightarrow V$ be a linear transformation. F is said to be invertible if it has an inverse that is, if there exists F^{-1} in $\mathcal{L}(V)$ such that $FF^{-1} = F^{-1}F = I$. On the other hand, F is invertible as a mapping if F is both one-one & onto. In such case F^{-1} also linear.

Problems:

- 1) P.T. the transformation $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is singular and find its Kernel if the transformation $F(a, b) = (2a - 4b, 3a - 6b)$

Soluⁿ: The matrix form of F is

$$F(a, b) = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

Now, the determinant of matrix is

$$\begin{vmatrix} 2 & -4 \\ 3 & -6 \end{vmatrix} = 2(-6) - (3)(-4) = 0$$

$\therefore$ the given transformation F is singular

* To find the Kernel

consider $F(a, b) = (0, 0)$

$$\Rightarrow (2a - 4b, 3a - 6b) = (0, 0)$$

$$2a - 4b = 0 \quad ; \quad 3a - 6b = 0$$

$$a - 2b = 0 \quad \quad a - 2b = 0$$

$$a = 2b \quad \quad a = 2b$$

Let $b = t$ (arbitrary)

$$\therefore a = 2t$$

$$\text{Hence } \text{Ker}(F) = \{(2t, t) \mid t \in \mathbb{R}\}$$

$$\text{equivalently } \text{Ker}(F) = \text{span}\{(2, 1)\}$$

2) Prove that $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ is singular & find its kernel.
if $T(x, y, z, t) = (2x, y, z+t)$

Soln: Let us consider

$$T(x, y, z, t) = (0, 0, 0)$$

$$\Rightarrow (2x, y, z+t) = (0, 0, 0)$$

$$2x=0, \quad y=0, \quad z+t=0$$

$$\Rightarrow x=0, \quad y=0, \quad z=-t \quad \text{--- (1)}$$

$$\text{Let } t=1, \Rightarrow z=-1$$

$$\text{then } (x, y, z, t) = (0, 0, -1, 1) \neq 0$$

$\therefore$ the transformation T is singular

Let $t=k$ (arbitrary)

$$\text{(1)} \Rightarrow z = -k$$

$$\therefore \text{Ker}(T) = \{(0, 0, -k, k) \mid k \in V(\mathbb{R})\}$$

3) Show that the linear map $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by
 $F(x, y) = (x-y, x-2y)$ is non-singular. Find its invertible.

Soln: Consider, $F(x, y) = (0, 0)$

$$\Rightarrow (x-y, x-2y) = (0, 0)$$

$$\Rightarrow x-y=0, \quad x-2y=0$$

$$\Rightarrow x=y; \quad x=2y$$

From both the above equations we have different values for x .

Hence the only solution possible is $x=0, y=0$

$\therefore$ The transformation F is non-singular.

Hence T is invertible.

Now, let $F(x, y) = (a, b)$, so that $(x, y) = F^{-1}(a, b)$

$$\Rightarrow (x-y, x-2y) = (a, b)$$

$$\Rightarrow \begin{matrix} x-y=a & \text{--- (1)} \\ x-2y=b & \text{--- (2)} \end{matrix}$$

Solve (1) & (2)

$$\begin{array}{rcl} \text{(1)} - \text{(2)} & ; & x-y=a \\ & & x-2y=b \\ \hline & (-) & (+) \quad (-) \\ & & y = a-b \end{array}$$

Sub y in (1)

$$x = a + y = a + (a-b)$$

$$\underline{x = 2a - b}$$

$$\text{Hence } F^{-1}(a, b) = (2a-b, a-b)$$

$$\text{or } F^{-1}(x, y) = (2x-y, x-y)$$

4) Let F be a linear transformation defined on a vector space $\mathbb{R}^2$ through $F(x, y) = (2x+y, 3x+2y)$. Show that F is invertible & hence find F^{-1} .

Soln: Consider $F(x, y) = (0, 0)$

$$\Rightarrow (2x+y, 3x+2y) = (0, 0)$$

$$\Rightarrow 2x+y=0, 3x+2y=0$$

The only ~~an~~ solution which satisfies both the equations is $x=0, y=0$

Hence the transformation F is non-singular
So that F is invertible.

$$\text{Let } F(x, y) = (a, b) \Rightarrow (x, y) = F^{-1}(a, b)$$

$$\Rightarrow (2x + y, 3x + 2y) = (a, b)$$

$$\begin{array}{rcl} 2x + y & = & a \\ 3x + 2y & = & b \end{array} \quad \text{--- (2)}$$

Solve (1) & (2)

$$\begin{array}{rcl} 2 \times (1) & - & (2) \\ \hline 4x + 2y & = & 2a \\ -3x - 2y & = & -b \\ \hline x & = & 2a - b \end{array}$$

$$\begin{array}{l} \text{From (1)} \quad y = a - 2x \\ y = a - 2(2a - b) \\ y = -3a + 2b \\ \underline{\underline{y = -3a + 2b}} \end{array}$$

$$\therefore F^{-1}(a, b) = (2a - b, -3a + 2b)$$

(OR)

$$F^{-1}(x, y) = (2x - y, -3x + 2y)$$

5) Let $G: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be defined by $G(x, y) = (x + y, x - 2y, 3x + y)$.
Show that G is non-singular & hence find its inverse.

Soln: Let $G(x, y) = (0, 0, 0)$

$$\Rightarrow (x + y, x - 2y, 3x + y) = (0, 0, 0)$$

$$\Rightarrow x + y = 0, \quad x - 2y = 0, \quad 3x + y = 0$$

The only solution which satisfy the above equations is
 $x = 0, y = 0$

$\therefore$ the given transformation is non-singular.

Since $\mathbb{R}^2$ & $\mathbb{R}^3$ having different dimensions.
the transformation G is not invertible.

$\Rightarrow G^{-1}$ does not exist.

6) Determine whether or not the linear map $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is non-singular. If not find its kernel, otherwise find its inverse. The mapping defined by

$$F(x, y, z) = (x+y+z, 2x+3y+5z, x+3y+7z)$$

Soln: Let $F(x, y, z) = (0, 0, 0)$

$$\Rightarrow (x+y+z, 2x+3y+5z, x+3y+7z) = (0, 0, 0)$$

$$\Rightarrow x+y+z=0, 2x+3y+5z=0, x+3y+7z=0$$

The matrix is

$$F = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 1 & 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 1 & 3 & 7 \end{vmatrix} = 1(21-15) - 1(14-5) + 1(6-3) = 6-9+3=0$$

$\therefore$ The transformation F is singular.

Now Consider,

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 1 & 3 & 7 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x+y+z=0 \quad \text{--- (1)}$$

$$y+3z=0 \quad \text{--- (2)}$$

Let $z=t$ (arbitrary)

$$\text{(2)} \Rightarrow y = -3t //$$

$$\text{(1)} \Rightarrow x = -y - z = -(-3t) - t = 2t$$

$$\therefore \text{Ker}(F) = \{(2t, -3t, t) / t \in \mathbb{R}\} \quad x=2t //$$

$$\text{equivalently } \text{Ker}(F) = \text{Span} \{(2, -3, 1)\}$$

Rank and Nullity of a Linear Transformation

Let $T: V \rightarrow W$ be a linear transformation. The range of T denoted by $R(T)$ is the set $R(T) = \{T(x) \mid x \in V\}$ which being the set of all images of the elements of V under the transformation T .

The dimension of $R(T)$ called the rank of T denoted $r(T)$. Further kernel of T denoted by $N(T)$ is the set

$$N(T) = \{x \in V \mid T(x) = 0\}$$

The dimension of $N(T)$ is called the Nullity of T denoted by $n(T)$

Rank-Nullity Theorem:

Let V be a finite dimensional vector space and $T: V \rightarrow W$ be a linear transformation. The theorem states that the sum of rank and nullity of T is equal to the dimension of V . i.e., $r(T) + n(T) = \dim(V)$

Working Procedure:

- Given a defined transformation say $T(x, y, z)$ we have to find $T(e_1), T(e_2), T(e_3)$ & write down the associated matrix $[A]$
- $[A]$ is reduced to the row echelon form. The number of non-zero rows is the rank and no. of zero rows is the nullity. The sum of these two should be equal to the dimension of the domain which verifies the associated theorem.
- Range space $R(T)$ will be the linear span of the non-zero rows in the row echelon form of A .
- Kernel of T : $N(T)$ will be the solution of the system of equations from $T(x, y, z) = 0$

* Find the Range, Null Space, Rank, Nullity in the case of the following linear transformations.
Also verify the rank-nullity theorem.

(1) $T: V_3(\mathbb{R}) \rightarrow V_2(\mathbb{R})$ defined by $T(x, y, z) = (y-x, y-z)$.

Soluⁿ: Given $T(x, y, z) = (y-x, y-z)$

$$T(e_1) = T(1, 0, 0) = (-1, 0)$$

$$T(e_2) = T(0, 1, 0) = (1, 1)$$

$$T(e_3) = T(0, 0, 1) = (0, -1)$$

$\therefore$ the coefficient matrix $[A] = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}$

$$A = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 + R_1$$

$$A \sim \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2$$

$$\sim \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

* Rank of $T = r(T) = 2$ - no. of non-zero rows

* Nullity of $T = n(T) = 1$ - no. of zero rows

$$r(T) + n(T) = 2 + 1 = 3 = \dim(V)$$

Hence the theorem is verified.

* $R(T) = L(S)$ being the linear span of $S = \{(-1, 0), (0, 1)\}$
 i.e., $\{x_1(-1, 0) + x_2(0, 1) = (-x_1, x_2)\}$

$\therefore R(T) = \{(-x_1, x_2) / x_1, x_2 \in \mathbb{R}\}$ is the range of T .

* To find $N(T)$, consider $T(x, y, z) = 0$

$$\Rightarrow (y-x, y-z) = (0, 0)$$

$$y-x=0, \quad y-z=0$$

$$\Rightarrow y=x, \quad y=z$$

$$\Rightarrow x=y=z$$

Let $z=t$ be an arbitrary.

$$\Rightarrow x=t=y$$

$\therefore N(T) = \{(t, t, t) / t \in \mathbb{R}\}$ is the null space (or) Kernel of T .

② $T: V_3(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ defined by

$$T(x, y, z) = (x+y, x-y, 2x+z)$$

$\rightarrow$ Given $T(x, y, z) = (x+y, x-y, 2x+z)$

$$T(e_1) = T(1, 0, 0) = (1, 1, 2)$$

$$T(e_2) = T(0, 1, 0) = (1, -1, 0)$$

$$T(e_3) = T(0, 0, 1) = (0, 0, 1)$$

$$\therefore \text{The matrix } [A] = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore R(T) = \text{Rank of } T = 3$$

$$n(T) = \text{Nullity of } T = 0$$

$$R(T) + n(T) = 3 + 0 = 3 = \dim[V_3(\mathbb{R})]$$

Hence the theorem is verified

$$* R(T) = L(S) = \{x_1(1, 1, 2) + x_2(0, 1, 1) + x_3(0, 0, 1)\}$$

$$\therefore R(T) = \{(x_1, x_1+x_2, 2x_1+x_2+x_3) / x_1, x_2, x_3 \in \mathbb{R}\} \text{ --- is the range of } T$$

$$* \text{ To find } N(T), \text{ consider } T(x, y, z) = 0$$

$$\Rightarrow (x+y, x-y, 2x+z) = (0, 0, 0)$$

$$x+y=0, \quad x-y=0, \quad 2x+z=0$$

-a -b -c

Solve (a), (b) & (c), we get

$$x=0, y=0, \text{ \& } z=0$$

$$\therefore x=y=z=0$$

$$\therefore N(T) = \{(0, 0, 0)\} \text{ is the null space or kernel of } T.$$

$$3) T: V_3(\mathbb{R}) \rightarrow V_3(\mathbb{R}) \text{ defined by } T(x, y, z) = (x+2y+z, z-x, y+z)$$

Soln : Given $T(x, y, z) = (x+2y+z, z-x, y+z)$

$$T(e_1) = T(1, 0, 0) = (1, -1, 0)$$

$$T(e_2) = T(0, 1, 0) = (2, 0, 1)$$

$$T(e_3) = T(0, 0, 1) = (1, 1, 1)$$

$\therefore$ The matrix $[A]$ is

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore r(T) = \text{Rank of } T = 2$$

$$\text{Nullity of } T = n(T) = 1$$

$$\therefore r(T) + n(T) = 2 + 1 = 3 = \dim[V_3(R)]$$

Hence the theorem is verified.

$$* R(T) = L(S) = \{x_1(1, -1, 0) + x_2(0, 2, 1)\}$$

$$\therefore R(T) = \{(x_1, -x_1 + 2x_2, x_2) \mid x_1, x_2, x_3 \in R\}$$

$$* \text{ To find } N(T), \text{ consider } T(x, y, z) = 0$$

$$\Rightarrow (x + 2y + z, z - x, y + z) = (0, 0, 0)$$

$$x + 2y + z = 0, \quad z - x = 0, \quad y + z = 0$$

$$x = z; \quad y = -z$$

Let z be a $z = t$ be an arbitrary.

$$\Rightarrow x = t, \quad y = -t$$

$$\therefore N(T) = \{(t, -t, t) \mid t \in R\} \text{ is the null space or kernel of } T$$

4) $T: V_3(\mathbb{R}) \rightarrow V_4(\mathbb{R})$ defined by

$$T(e_1) = (0, 1, 0, 2), T(e_2) = (0, 1, 1, 0), T(e_3) = (0, 1, -1, 4)$$

$\Rightarrow$ Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 4 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + R_2$$

$$\sim \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \text{Rank of } T = \lambda(T) = 2$$

$$\text{Nullity of } T = n(T) = 1$$

$$\lambda(T) + n(T) = 2 + 1 = 3 = \dim(V_3(\mathbb{R}))$$

Hence the theorem is verified.

$$* R(T) = \mathcal{L}(S) = \{ x_1(0, 1, 0, 2) + x_2(0, 0, 1, -2) \}$$

$$R(T) = \{ (0, x_1, x_2, 2x_1 - 2x_2) / x_1, x_2 \in \mathbb{R} \}$$

* To find $N(T)$, consider $T(x, y, z) = 0$

$$\Rightarrow T(xe_1 + ye_2 + ze_3) = (0, 0, 0, 0)$$

$$xT(e_1) + yT(e_2) + zT(e_3) = (0, 0, 0, 0)$$

$$x(0, 1, 0, 2) + y(0, 1, 1, 0) + z(0, 1, -1, 4) = (0, 0, 0, 0)$$

$$(0, x + y + z, y - z, 2x + 4z) = (0, 0, 0, 0)$$

$$\Rightarrow x + y + z = 0, y - z = 0, 4x + 2z = 0$$

$$y = z$$

$$2x = -4z \Rightarrow x = -2z$$

Let $z = t$ be an arbitrary; $\therefore y = t, x = -2t$.

$$\therefore N(T) = \{ (-2t, t, t) / t \in \mathbb{R} \} \text{ is the kernel of } T.$$

5) $T: V_2(\mathbb{R}) \rightarrow V_3(\mathbb{R})$ defined by $T(1, 2) = (15, -20, 5)$ and $T(2, 1) = (12, -16, 4)$

$\Rightarrow$ Soln: We have $(1, 2) = 1(1, 0) + 2(0, 1)$

$$T(1, 2) = 1 \cdot T(e_1) + 2 \cdot T(e_2)$$

$$\Rightarrow T(e_1) + 2T(e_2) = (15, -20, 5) \quad \text{--- (1)}$$

$$\& \quad 2T(e_1) + T(e_2) = (12, -16, 4) \quad \text{--- (2)}$$

$$2 \times (1) - (2) \Rightarrow 2T(e_1) + 4T(e_2) = (30, -40, 10)$$

$$\begin{array}{r} 2T(e_1) + 4T(e_2) = (30, -40, 10) \\ \underline{2T(e_1) + T(e_2) = (12, -16, 4)} \\ 3T(e_2) = (18, -24, 6) \end{array}$$

$$3T(e_2) = (18, -24, 6)$$

$$T(e_2) = (6, -8, 2)$$

$$(2) \Rightarrow 2T(e_1) + (6, -8, 2) = (12, -16, 4)$$

$$2T(e_1) = (12, -16, 4) - (6, -8, 2) = (6, -8, 2)$$

$$\Rightarrow T(e_1) = (3, -4, 1)$$

$\therefore$ The matrix $[A]$ is

$$A = \begin{bmatrix} 3 & -4 & 1 \\ 6 & -8 & 2 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - R_1$$

$$A = \begin{bmatrix} 3 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore r(T) = 1 = \text{rank of } T$$

$$n(T) = 1 = \text{nullity of } T$$

$$r(T) + n(T) = 1 + 1 = 2 = \dim(V_2(\mathbb{R}))$$

Hence the theorem is verified.

$$* R(T) = L(S) = \{x_1(3, -4, 1)\}$$

$\therefore R(T) = \{(3x_1, -4x_1, x_1) / x_1 \in \mathbb{R}\}$ is the range of T .

* ~~A~~ To find null space, consider $T(x, y) = 0$

$$\rightarrow T[xe_1 + ye_2] = (0, 0, 0)$$

$$\rightarrow xT(e_1) + yT(e_2) = (0, 0, 0)$$

$$x(3, -4, 1) + y(6, -8, 2) = (0, 0, 0)$$

$$(3x + 6y, -4x - 8y, x + 2y) = (0, 0, 0)$$

$$3x + 6y = 0; -4x - 8y = 0, x + 2y = 0$$

$$\div 3 \Rightarrow x + 2y = 0, \div 4 \Rightarrow x + 2y = 0, x + 2y = 0$$

Equivalently we have $x + 2y = 0$

if $y = t$ be an arbitrary

$$\text{then } x = -2t$$

$\therefore N(T) = \{(-2t, t) / t \in \mathbb{R}\}$ is the kernel of T or null space.