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DEPARTMENT OF MATHEMATICS

Mathematics for Computer Science

(BCS301)

MODULE 1 : PROBABILITY DISTRIBUTIONS

(As per Visvesvaraya Technological University Syllabus 2022 Scheme)

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Module - 01

Probability Distribution

⇒ **Random Variables**

In a random experiment if a real variable is associated with every outcome then it is called a random variable or stochastic variable. It is denoted by X, Y, Z... The set of all real numbers of a random variable X is called the range of X

$X : S \to R$

$S \to$ Sample space

Example :

$X \to$ number of heads

$Y \to$ number of tails

Outcome :	HH	HT	TH	TT
(R.V) X :	2	1	1	0
Y :	0	1	1	2

⇒ **Discrete and Continuous Random variables**

If a random variables takes finite or countably infinite numbers of values then it is called discrete random variables

Example : Tossing a coin and observing outcomes

If a random variables takes non countable infinite numbers of values then it is called a non discrete or continuous random variable. It can assume any value in the interval of a real numbers

Example : Weight of article

⇒ **Discrete probability distribution**

If for each value x_i of a discrete random variable X, a real no $p(x_i)$ is assigned such that

(i) $p(x_i) \geq 0$ (ii) $\sum p(x_i) = 1$

then the function $p(x)$ is called probability function

If the probability that X take the values x_i is p_i, then

$p(X = x_i) = p_i$ or $p(x_i)$

The set of values $[x_i, p(x_i)]$ is called discrete (finite) probability distribution of discrete random values 'X'. The function $p(x)$ is called probability mass function

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The distribution function $f(x)$ defined by $f(x) = p(X \leq x) = \sum_{i=1}^{x} p(x_i)$, x being an integer is called cummulative distribution function (c.d.f). For discrete probability distribution,

$$\text{Mean}(\mu) = \sum_i x_i p(x_i)$$

$$\text{Variance}(V) = \sum_i (x_i - \mu)^2 \cdot p(x_i)$$

$$\text{Standard deviation}(\sigma) = \sqrt{V}$$

$$[\text{Or Variance} = \sum x_i^2 p(x_i) - [\sum_i x_i p(x_i)]^2$$

Note

1. $0 \leq P(A) \leq 1$
2. $P(\bar{A}) = 1 - P(A)$

⇒ Problems

***
1. Find mean and variance given the following table which represents probability distribution of a finite random variable X, also determine standard deviation
6M

x_i :	-2	-1	0	1	2	3
$p(x_i)$:	0.1	K	0.2	2K	0.3	K

sol (i) $p(x_i) \geq 0$
(ii) $\Sigma p(x_i) = 1$

$p(x_i)$ is a probability mass function

$0.1 + K + 0.2 + 2K + 0.3 + K = 1$
$0.6 + 4K = 1$
$4K = 1 - 0.6$
$4K = 0.4$
$\boxed{K = 0.1}$

probability distribution is

x_i	-2	-1	0	1	2	3
$p(x_i)$	0.1	0.1	0.2	0.2	0.3	0.1
$x_i\, p(x_i)$	-0.2	-0.1	0	0.2	0.6	0.3
$(x_i - 0.8)^2 p(x_i)$	0.784	0.324	0.128	0.008	0.432	0.484

Mean $= \Sigma x_i p(x_i)$
Mean $(\mu) = -0.2 - 0.1 + 0 + 0.2 + 0.6 + 0.3$
$\boxed{\text{Mean } (\mu) = 0.8}$

Variance $(V) = \Sigma (x_i - \mu)^2 p(x_i)$
Variance $(V) = 0.784 + 0.324 + 0.128 + 0.008 + 0.432 + 0.484$

$\boxed{\text{Variance } (V) = 2.16}$

Standard deviation $(\sigma) = \sqrt{V}$

$S.D(\sigma) = \sqrt{2.16}$

$\boxed{S.D(\sigma) = 1.4697}$

2. Find the mean and variance for the following probability distribution using suitable K value

x_i:	0	1	2	3	4	5
$p(x_i)$:	K	5K	10K	10K	5K	K

(i) $p(x_i) \geq 0$

(ii) $\Sigma p(x_i) = 1$

$p(x_i)$ is a probability mass function

$K + 5K + 10K + 10K + 5K + K = 1$

$32K = 1$

$\boxed{K = \frac{1}{32}}$

probability distribution is

x_i:	0	1	2	3	4	5
$p(x_i)$:	$1/32$	$5/32$	$5/16$	$5/16$	$5/32$	$1/32$
$x_i p(x_i)$:	0	$5/32$	$5/8$	$15/16$	$5/8$	$5/32$
$(x_i - 2.5)^2 p(x_i)$:	$25/128$	$45/128$	$5/64$	$5/64$	$45/128$	$25/128$

Mean (μ) : $\Sigma x_i p(x_i)$

Mean $(\mu) = 0 + \frac{5}{32} + \frac{5}{8} + \frac{15}{16} + \frac{5}{8} + \frac{5}{32}$

$\boxed{\text{Mean } (\mu) = 2.5}$

Variance $= \Sigma (x_i - 2.5)^2 p(x_i)$

Variance $(V) = \frac{25}{128} + \frac{45}{128} + \frac{5}{64} + \frac{5}{64} + \frac{45}{128} + \frac{25}{128}$

$\boxed{\text{Variance } (V) = 1.25}$

MQP/Dec 2023

*** 3. The probability distribution of finite random variable X is given by

MQP/Dec 2023

x:	0	1	2	3	4	5	6	7
$P(x)$:	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

$K=?$, $P(x<6)$, $P(x\geq 6)$, $P(3<x\leq 6)$, $P(0<x<5)$

sol $p(x)$ is p.m.f

(i) $p(x)\geq 0$
(ii) $\Sigma p(x)=1$

$0+K+2K+2K+3K+K^2+2K^2+7K^2+K=1$
$10K^2+9K=1$
$10K^2+9K-1=0$
$K=0.1$ or $K=-1$
[$K=-1$, $p(x)\geq 0$] {so $K=-1$ is not possible}

$\boxed{K=0.1}$

probability distribution is:

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

(i) $K=0.1$
(ii) $p(x<6) = p(x=0)+p(x=1)+p(x=2)+p(x=3)+p(x=4)+p(x=5)$
$= 0+0.1+0.2+0.2+0.3+0.01$
$\boxed{p(x<6)=0.81}$ (or)

$p(x<6)=1-p(x\geq 6)$
$p(x<6)=1-[p(x=6)+p(x=7)]$

$p(x<6) = 1-[0.02+0.17]$

$\boxed{p(x<6) = 0.81}$

$p(x \geq 6) = p(x=6) + p(x=7)$

$p(x \geq 6) = 0.02 + 0.17$

$\boxed{p(x \geq 6) = 0.19}$

$p(3<x \leq 6) = p(4) + p(5) + p(6)$

$p(3<x \leq 6) = 0.3 + 0.01 + 0.02$

$\boxed{p(3<x \leq 6) = 0.33}$

$p(0<x<5) = p(x=1) + p(x=2) + p(x=3) + p(x=4)$

$p(0<x<5) = 0.1 + 0.2 + 0.2 + 0.3$

$\boxed{p(0<x<5) = 0.80}$

* MQP

4. The p.m.f of x is given by

MQP

x:	0	1	2	3	4	5	6
$p(x)$:	K	3K	5K	7K	9K	11K	13K

Find K, $P(x \geq 5)$, $p(3<x \leq 6)$, $P(x<4)$, $P(3<x<5)$, $P(x>1)$

Also find mean of x or $E(X)$ [Exception]

sol $p(x)$ is p.m.f

(i) $p(x) \geq 0$

(ii) $\Sigma p(x) < 1$

$K + 3K + 5K + 7K + 9K + 11K + 13K = 1$

$49K = 1$

$\boxed{K = \frac{1}{49}}$

probability distribution is

x_i:	0	1	2	3	4	5	6
$p(x_i)$:	$1/49$	$3/49$	$5/49$	$7/49$	$9/49$	$11/49$	$13/49$
$x_i p(x_i)$	0	$3/49$	$10/49$	$3/7$	$36/49$	$55/49$	$78/49$

$P(x \geq 5) = P(x=5) + P(x=6)$

$P(x \geq 5) = \frac{11}{49} + \frac{13}{49}$

$\boxed{P(x \geq 5) = \frac{24}{49} = 0.4898}$

$P(3 < x \leq 6) = P(x=4) + P(x=5) + P(x=6)$

$P(3 < x \leq 6) = \frac{9}{49} + \frac{11}{49} + \frac{13}{49}$

$\boxed{P(3 < x \leq 6) = \frac{33}{49} = 0.6735}$

$P(x<4) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$P(x<4) = \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49}$

$\boxed{P(x<4) = \frac{16}{49} = 0.3265}$

$P(3<x<6) = P(x=4) + P(x=5)$

$P(3<x<6) = \frac{9}{49} + \frac{11}{49}$

$\boxed{P(3<x<6) = \frac{20}{49} = 0.4082}$

$P(x>1) = P(x=2) + P(x=3) + P(x=4) + P(x=5) + P(x=6)$

$P(x>1) = \frac{5}{49} + \frac{7}{49} + \frac{9}{49} + \frac{11}{49} + \frac{13}{49}$

$\boxed{P(x>1) = \frac{45}{49} = 0.9184}$

$\text{Mean}(\mu) = 0 + \frac{3}{49} + \frac{10}{49} + \frac{3}{7} + \frac{36}{49} + \frac{55}{49} + \frac{78}{49}$

$\text{Mean}(\mu) = 29/7$

$\boxed{\text{Mean}(\mu) = 4.1428}$

5. If the random variable X takes the values 1, 2, 3, 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$ find p.d.f and c.d.f (cumulative distribution function) of X

Sol

X :	1	2	3	4
p.d.f $P(x)$:	P_1	P_2	P_3	P_4

(i) $P(x) \geq 0$

(ii) $\Sigma P(x) = 1$

$P_1 + P_2 + P_3 + P_4 = 1 \rightarrow (1)$

Given,

$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$

$2P_1 = 3P_2 = P_3 = 5P_4$

$2P_1 = 3P_2$ | $2P_1 = P_3$ | $2P_1 = 5P_4$

$\boxed{P_2 = \frac{2P_1}{3}}$ | $\boxed{P_3 = 2P_1}$ | $\boxed{P_4 = \frac{2P_1}{5}}$

(1) $\Rightarrow$

$P_1 + \frac{2P_1}{3} + 2P_1 + \frac{2P_1}{5} = 1$

$P_1 [1 + \frac{2}{3} + 2 + \frac{2}{5}] = 1$

$P_1 [\frac{61}{15}] = 1$

$\boxed{P_1 = \frac{15}{61}}$

$P_2 = \frac{2P_1}{3}$

$P_2 = \frac{2}{3} \times \frac{15}{61}$

$\boxed{P_2 = \frac{10}{61}}$

$P_3 = 2P_1$

$P_3 = 2 \times \frac{15}{61}$

$\boxed{P_3 = \frac{30}{61}}$

$P_4 = \frac{2}{5} P_1$

$P_4 = \frac{2}{5} \times \frac{15}{61}$

$\boxed{P_4 = \frac{6}{61}}$

	X	1	2	3	4
p.d.f	$P(x)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$
c.d.f	$F(x)$	$\frac{15}{61}$	$\frac{25}{61}$	$\frac{55}{61}$	$\frac{61}{61}=1$

6. Find the value of K such that $P(x) = \frac{K}{2^x}$ where $x = 1, 2, 3, \ldots$, represents probability distribution

sol

(i) $p(x) \geq 0$

(ii) $\sum p(x) = 1$

$$\sum_{x=1}^{\infty} \frac{K}{2^x} = 1$$

$$K \sum_{x=1}^{\infty} \frac{1}{2^x} = 1$$

It is geometric progression

$$K\left\{\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} \ldots\right\} = 1$$

$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$ is a GP

$a = \frac{1}{2}$, $r = \frac{1}{2}$

$S_\infty = \frac{a}{1-r} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$

$$K\{1\} = 1$$

$\boxed{K = 1}$

7. A random variable X takes the values $-3, -2, -1, 0, 1, 2, 3$ such that $P(x=0) = P(x<0)$ and $P(X=-3) = P(X=-2) = P(X=-1) = P(x=1) = P(X=2) = P(X=3)$ find the probability distribution

sol Let the probability distribution $x, p(x)$ be

x :	-3	-2	-1	0	1	2	3
$p(x)$:	P_1	P_2	P_3	P_4	P_5	P_6	P_7

$P(x=0) = P(x<0)$

$$\boxed{P_4 = P_1 + P_2 + P_3} \rightarrow (1)$$

$P(x=-3) = P(x=-2) = P(x=-1) = P(x=1) = P(x=2) = P(x=3)$

$$\boxed{P_1 = P_2 = P_3 = P_5 = P_6 = P_7} \rightarrow (2)$$

Substitute (2) in (1)

$P_4 = P_1 + P_1 + P_1$

$$\boxed{P_4 = 3P_1} \rightarrow (3)$$

W.K.T $\Sigma P(x) = 1$

$P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_7 = 1 \rightarrow (4)$

$P_4 + P_4 + P_5 + P_6 + P_7 = 1$ [from eqn (1)]

$2P_4 + 3P_1 = 1$ [from (2)]

$2(3P_1) + 3P_1 = 1$

$9P_1 = 1$

$$\boxed{P_1 = \frac{1}{9}}$$

$P_1 = P_2 = P_3 = P_5 = P_6 = P_7$

$P_4 = 3P_1$

$$\boxed{P_4 = 3 \times \frac{1}{9}}$$

$$\boxed{P_4 = \frac{1}{3}}$$

x	-3	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

8. The p.d.f of a random variable x is given by the following table

x	-3	-2	-1	0	1	2	3
$P(x)$	K	2K	3K	4K	3K	2K	K

Find (i) The value of K
(ii) $P(x>1)$
(iii) $P(-1<x<2)$
(iv) Mean of x
(v) S.D of x

sol $P(x)$ is a p.d.f
(i) $p(x) \geq 0$
(ii) $\Sigma P(x) = 1$

(i) $K + 2K + 3K + 4K + 3K + 2K + K = 1$
$16K = 1$
$\boxed{K = \frac{1}{16}}$

Let the probability distribution be

x_i :	-3	-2	-1	0	1	2	3
$P(x_i)$:	1/16	1/8	3/16	1/4	3/16	1/8	1/16
$x_i P(x_i)$:	-3/16	-1/4	-3/16	0	3/16	1/4	3/16
$(x_i - 0)^2 P(x_i)$:	9/16	1/2	3/16	0	3/16	1/2	9/16

(ii) $P(x>1) = P(x=2) + P(x=3)$
$P(x>1) = \frac{1}{8} + \frac{1}{16}$

$$\boxed{P(x>1) = \frac{3}{16} = 0.1875}$$

(iii) $P(-1<x<2) = P(x=0)+P(x=1)$

$P(-1<x<2) = \frac{1}{4} + \frac{3}{16}$

$$\boxed{P(-1<x<2) = \frac{7}{16} = 0.4375}$$

(iv) Mean (μ) $= \Sigma P(xi).xi$

Mean (μ) $= \frac{-3}{16} - \frac{1}{4} \; \frac{-3}{16} + 0 + \frac{3}{16} + \frac{1}{4} + \frac{3}{16}$

$$\boxed{\text{Mean}(\mu) = 0}$$

(v) Variance (v) $= \Sigma (xi-0)^2 P(xi)$

Variance (v) $= \frac{9}{16} + \frac{1}{2} + \frac{3}{16} + 0 + \frac{3}{16} + \frac{1}{2} + \frac{9}{16}$

$$\boxed{\text{Variance (V)} = \frac{5}{2} = 2.5}$$

S.D (σ) $= \sqrt{V}$

S.D (σ) $= \sqrt{2.5}$

$$\boxed{S.D(\sigma) = 1.5811}$$

9. The p.d.f of a random variable x is given by the following table

x	-2	-1	0	1	2	3	4
P(x)	0.1	0.1	K	0.1	2K	K	K

Find (i) The value of K
(ii) Mean or E(x)
(iii) S.D

sol $P(xi) \geq 0$ p(xi) is a p.d.f

$\Sigma P(xi) = 1$

$0.1 + 0.1 + K + 0.1 + 2K + K + K = 1$

$0.3 + 5K = 1$

$5K = 1 - 0.3$

$5K = 0.7$

(i) $\boxed{K = 0.14}$

x_i	-2	-1	0	1	2	3	4
$P(x_i)$	0.1	0.1	0.14	0.1	0.28	0.14	0.14
$x_i P(x_i)$	-0.2	-0.1	0	0.1	0.56	0.42	0.56
$(x_i - 1.34)^2 P(x_i)$	1.1156	0.5476	0.2514	0.0116	0.1220	0.3858	0.9906

(ii) Mean $(\mu) = \Sigma x_i P(x_i)$

Mean $(\mu) = -0.2 - 0.1 + 0 + 0.1 + 0.56 + 0.42 + 0.56$

$\boxed{\text{Mean } (\mu) = 1.34}$

(iii) Variance $(V) = \Sigma (x_i - 1.34)^2 P(x_i)$

Variance $(V) = 1.1156 + 0.5476 + 0.2514 + 0.0116 + 0.1220 + 0.3858 + 0.9906$

$\boxed{\text{Variance } (V) = 3.4246}$

S.D. $(\sigma) = \sqrt{V}$

S.D $(\sigma) = \sqrt{3.4256}$

$\boxed{\text{S.D } (\sigma) = 1.8506}$

*** MQP

* 10. Determine the value of K, so that the function
MQP $f(x) = K(x^2+4)$ for $x = 0, 1, 2, 3$ can serve as a probability distribution of the discrete random variable X. Also find

(i) $P(0 < x \leq 2)$

(ii) $P(x \geq 1)$

Sol (i) $f(x_i) \geq 0$

(ii) $\Sigma f(x_i) = 1$

$$\sum_{x=0}^{3} K(x^2+4)=1$$

$$K\sum_{x=0}^{3}(x^2+4)=1$$

$$K\{4+5+8+13\}=1$$

$$K(30)=1$$

$$30K=1$$

$$\boxed{K=\frac{1}{30}}$$

x_i	0	1	2	3
$P(x_i)$	$2/15$	$1/6$	$4/15$	$13/30$

$=1$

(i) $P(0<x\le 2)=P(x=1)+P(x=2)$

$P(0<x\le 2)=\frac{1}{6}+\frac{4}{15}$

$$\boxed{P(0<x\le 2)=\frac{13}{30}=0.4333}$$

(ii) $P(x\ge 1)=P(x=1)+P(x=2)+P(x=3)$

$P(x\ge 1)=\frac{1}{6}+\frac{4}{15}+\frac{13}{30}$

$$\boxed{P(x\ge 1)=\frac{13}{15}=0.8667}$$

*** 11. MQP A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers find the probability distribution for the number of defectives. Find the mean and variance of the distribution

MQP

sol Let X be a random variable
$X = \{0, 1, 2\}$

The total number of microcomputer = 8
Number of defective microcomputer = 3
Number of non-defective microcomputer = 5

⇒ Case 1: probability of selecting non defective microcomputer $(X=0)$

$$\boxed{P(X=0) = \frac{5C_2}{8C_2} = \frac{5}{14}}$$

⇒ Case 2: probability of selecting one defective micro-computer $(X=1)$

$$\boxed{P(X=1) = \frac{3C_1 \times 5C_1}{8C_2} = \frac{15}{28}}$$

⇒ Case 3: probability of selecting two defective micro-computer $(X=2)$

$$\boxed{P(X=2) = \frac{3C_2 \times 5C_0}{8C_2} = \frac{3}{28}}$$

$X = x$	0	1	2	
$p(X=x)$	5/14	15/28	3/28	=1
$\Sigma p(x_i) x_i$	0	15/28	3/14	
$(x_i - 0.75)^2 p(x_i)$	0.2009	0.0335	0.1674	

Mean $(\mu) = \Sigma x_i p(x_i)$
Mean $(\mu) = 0 + \frac{15}{28} + \frac{3}{14}$

$$\boxed{\text{Mean } (\mu) = 3/4 = 0.75}$$

Variance $= \Sigma (x_i - 0.75)^2 p(x_i)$
Variance $= 0.2009 + 0.0335 + 0.1674$
$\boxed{\text{Variance} = 0.4018}$

⇒ **Bernoulli trial**

A random experiment with only 2 possible outcomes categorized as success and failure is called Bernoulli trial where the probability of success (p) is same for each trial and probability of failure (q) is 1-p

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⇒ **Binomial distribution**

If p is the probability of success and q is the probability of failure, the probability of x successes out of x trial is given by

$$p(x) = {}^nC_x \, p^x q^{n-x}$$

Example :- Tossing a coin 3 times and probability of 2 heads

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$P(X = 2 \text{ heads}) = \frac{3}{8}$

$n = 3 \quad x = 2 \quad p = P(H) = \frac{1}{2}, \; q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

$$p(x=2) = 3C_2 \, (0.5)^2 (0.5)^{3-2} = \frac{3}{8}$$

probability distribution of $(x, p(x))$ is given by

x:	0	1	2	.... n
$p(x)$:	q^n	${}^nC_1 p^1 q^{n-1}$	${}^nC_2 p^2 q^{n-2}$	... p^n

(i) $p(x) \geq 0$

(ii) $\Sigma p(x) = q^n + {}^nC_1 p^1 q^{n-1} + {}^nC_2 p^2 q^{n-2} + \ldots + p^n$

$= (q+p)^n = 1^n = 1$

$\Sigma p(x) = 1$

$(x, p(x))$ is probability distribution

MQP * ⇒ **Mean and standard deviation of binomial distribution**

7M (MQP)

Dec 2023

Mean $(\mu) = \Sigma x\, p(x)$

Mean $(\mu) = \sum_{x=0}^{n} x\ {}^{n}C_{x}\ p^{x} q^{n-x}$

Mean $(\mu) = \sum_{x=0}^{n} x \left[\frac{n!}{(n-x)!\,x!}\right] p^{x} q^{n-x}$

Mean $(\mu) = \sum_{x=0}^{n} \cancel{x} \left[\frac{n(n-1)!}{(n-x)!\,\cancel{x}(x-1)!}\right] p^{x+1-1} q^{n-x}$

Mean $(\mu) = \sum_{x=1}^{n} \frac{n(n-1)!}{(n-x)!\,(x-1)!}\ p^{1} p^{x-1} q^{n-x}$

Mean $(\mu) = np \sum_{x=1}^{n} \frac{(n-1)!}{(n-x)!\,(x-1)!}\ p^{x-1} q^{n-x}$

Mean $(\mu) = np \sum_{x=1}^{n} \frac{(n-1)!}{(n-x+1-1)\,(x-1)!}\ p^{x-1} q^{n-x+1-1}$

Mean $(\mu) = np \sum_{x=1}^{n} \frac{(n-1)!}{((n-1)-(x-1))!\,(x-1)!}\ p^{x-1} q^{[(n-1)-(x-1)]}$

Mean $(\mu) = np \sum_{x=1}^{n} {}^{n-1}C_{x-1}\ p^{x-1} q^{[(n-1)-(x-1)]}$

Mean $(\mu) = np\,(1)$

$$\boxed{\text{Mean } (\mu) = np}$$

$$\text{Variance} = \Sigma x^2 p(x) - [\Sigma x p(x)]^2$$

$$\text{Variance} = \Sigma x^2 p(x) - (np)^2 \rightarrow (1)$$

Consider,

$$\Sigma x^2 p(x) = \Sigma (x^2 - x + x)\, p(x)$$

$$\Sigma x^2 p(x) = \Sigma (x^2 - x) p(x) + \Sigma x p(x)$$

$$\Sigma x^2 p(x) = \Sigma\, x(x-1)\, p(x) + np$$

$$\Sigma x^2 p(x) = \sum_{x=0}^{n} x(x-1)\, {}^{n}c_{x}\, p^{x} q^{n-x} + np$$

$$\Sigma x^2 p(x) = \sum_{x=0}^{n} x(x-1) \frac{n!}{(n-x)!\,x!} p^{x} q^{n-x} + np$$

$$\Sigma x^2 p(x) = \sum_{x=0}^{n} \frac{\cancel{x(x-1)}\, n(n-1)(n-2)!}{(n-x)!\,\cancel{x(x-1)}(x-2)!} p^{x} q^{n-x} + np$$

$$\Sigma x^2 p(x) = \sum_{x=2}^{n} \frac{n(n-1)(n-2)!}{(n-x)!(x-2)!} p^{x} q^{n-x} + np$$

$$\Sigma x^2 p(x) = n(n-1) \sum_{x=2}^{n} \frac{(n-2)!}{(n-x+2-2)!\,(x-2)!} p^{x+2-2} q^{(n-x+2-2)} + np$$

$$\Sigma x^2 p(x) = n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{[(n-2)-(x-2)]!\,(x-2)!} p^{x-2} q^{[(n-2)-(x-2)]} + np$$

$$\Sigma x^2 p(x) = n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}c_{x-2}\, p^{x-2} q^{[(n-2)-(x-2)]} + np$$

$$\Sigma x^2 p(x) = n(n-1)p^2 (1) + np \rightarrow (2)$$

Substitute (2) in (1)

$$\text{Variance} = n(n-1)p^2 + np - n^2p^2$$

$$\text{Variance} = \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$\text{Variance} = np - np^2$$

$$\text{Variance} = np(1-p)$$

$$\boxed{\text{Variance} = npq}$$

$$S.D = \sqrt{V} \Rightarrow \boxed{S.D = \sqrt{npq}}$$

⇒ **Poisson distribution**

This is regarded as the limiting form of the binomial distribution when n is very large ($n \to \infty$) and p, probability of success is very small ($p \to 0$) so that np tends to finite fixed constant say 'm'

$$\boxed{p(x) = \frac{m^x e^{-m}}{x!}}$$

is known as poisson distribution function of the random variable 'x', Here the 'x' is called poisson variate

---

poisson distribution is given by

x	0	1	2	3	----
$p(x)$	e^{-m}	me^{-m}	$\frac{m^2 e^{-m}}{2!}$	$\frac{m^3 e^{-m}}{3!}$	----

(i) $p(x) \geq 0$

(ii) $\Sigma p(x) = e^{-m} + me^{-m} + \frac{m^2 e^{-m}}{2!} + \frac{m^3 e^{-m}}{3!} + ---$

$$\Sigma p(x) = e^{-m}\left[1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + --\right]$$

$$\Sigma p(x) = e^{-m} \cdot e^{m}$$

$$\Sigma p(x) = e^{0}$$

$$\boxed{\Sigma p(x) = 1}$$

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** ⇛
7M (MQP)

Mean and Standard deviation of poisson distribution

Mean $= \Sigma x\, p(x)$

Mean $= \sum_{x=0}^{\infty} x \dfrac{m^x e^{-m}}{x!}$

Mean $= \sum_{x=0}^{\infty} \cancel{x} \dfrac{m^{x+1-1} e^{-m}}{\cancel{x}(x-1)!}$

Mean $= \sum_{x=1}^{\infty} \dfrac{m^{x-1} m^1 e^{-m}}{(x-1)!}$

Mean $= m e^{-m} \sum_{x=1}^{\infty} \dfrac{m^{x-1}}{(x-1)!}$

Mean $= m e^{-m} (e^m)$

Mean $= m (e^{-m} e^m)$

Mean $= m (1)$

$\boxed{\text{Mean} = m}$

Variance $= \Sigma x^2 p(x) - [\Sigma x\, p(x)]^2$

Variance $= \Sigma x^2 p(x) - m^2 \rightarrow (1)$

Consider, $\Sigma x^2 p(x) = \Sigma (x^2 - x + x)\, p(x)$

$\Sigma x^2 p(x) = \Sigma (x^2 - x)\, p(x) + \Sigma x\, p(x)$

$\Sigma x^2 p(x) = \sum_{x=0}^{\infty} \dfrac{\cancel{x(x-1)}\, m^x e^{-m}}{\cancel{x(x-1)}(x-2)!} + m$

$\Sigma x^2 p(x) = \sum_{x=2}^{\infty} \dfrac{m^{x+2-2} e^{-m}}{(x-2)!} + m$

$\Sigma x^2 p(x) = m^2 e^{-m} \sum_{x=2}^{\infty} \dfrac{m^{x-2}}{(x-2)!} + m$

$$\Sigma x^2 p(x) = m^2 e^{-m} \{1 + \frac{m}{1} + \frac{m^2}{2!} + \cdots\} + m$$

$$\Sigma x^2 p(x) = m^2 e^{-m} e^{m} + m$$

$$\Sigma x^2 p(x) = m^2 e^{0} + m$$

$$\Sigma x^2 p(x) = m^2 + m \rightarrow (2)$$

Substitute (2) in (1)

$$\text{Variance} = m^2 + m - m^2$$

$$\boxed{\text{Variance} = m}$$

$$\text{Standard deviation} = \sqrt{V}$$

$$\boxed{\text{standard deviation} = \sqrt{m}}$$

⇒ problems

1. The probability of germination of a seed in a packet of seeds is found to be 0.7. If 10 seeds are taken for experimenting on germination in a lab find probability that

(i) 8 seeds germinate

(ii) atleast 8 seeds germinate

$p = 0.7$, $n = 10$, $q = 1 - p = 1 - 0.7 = 0.3$

By binomial distribution,

$$p(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$p(x) = 10C_{x} (0.7)^{x} (0.3)^{10-x}$$

(i) $P(x = 8) = 0.2335$

(ii) $p(x \geq 8) = p(x=8) + p(x=9) + p(x=10)$

$p(x \geq 8) = 0.2335 + 0.1211 + 0.0282$

$\boxed{p(x \geq 8) = 0.3828}$

*** 2. A die is thrown 8 times find the probability 3 falls

7M

(i) Exactly 2 times

(ii) Atleast once

(iii) At the most 7 times

$p = \frac{1}{6}$, $n = 8$, $q = 1 - \frac{1}{6} = \frac{5}{6}$

By binomial distribution

$p(x) = nc_x \, p^x \, q^{n-x}$

$$\boxed{p(x) = 8c_x \left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{8-x}}$$

(i) $\boxed{p(x=2) = 0.2605}$

(ii) $p(x \geq 1) = 1 - [p(x < 1)]$

$p(x \geq 1) = 1 - [0.2326]$

$\boxed{p(x \geq 1) = 0.7674}$

(iii) $p(x \leq 7) = 1 - [p(x > 7)]$

$p(x \leq 7) = 1 - [p(x=8)]$

$p(x \leq 7) = 1 - 0$

$\boxed{p(x \leq 7) = 1}$

* 3. 'X' is a binomially distributed random variable if mean and variance of 'x' are 2 and $\frac{3}{2}$ respectively find the distribution function and binomial distribution

7M

sol

Mean $= 2$

$np = 2$

Variance $= \frac{3}{2}$

$npq = \frac{3}{2}$

$2q = \frac{3}{2}$

$\boxed{q = \frac{3}{4}}$

$p = 1 - q = 1 - \frac{3}{4}$

$\boxed{p = \frac{1}{4}}$

$np = 2$

$\frac{n}{4} = 2$

$\boxed{n = 8}$

Distribution function

$p(x) = n_{C_x} p^x q^{n-x}$

$\boxed{p(x) = 8_{C_x} \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{8-x}}$

Binomial distribution,

x	0	1	2	3	4	5
$p(x)$	0.1001	0.2670	0.3115	0.2076	0.0865	0.0231

6	7	8
0.0038	0.0004	0

4. In a quiz contest of answering 'YES' or 'NO' what is the probability of guessing atleast six(6) answers correct out of 10 questions asked, also find the probability of the same if there are 4 options for a correct answer

sol $n = 10$ $\quad p = \frac{1}{2}$ $\quad q = 1 - p = \frac{1}{2}$

By Binomial distribution

$p(x) = {}^{n}c_{x}\, p^{x}\, q^{n-x}$

$p(x) = {}^{10}c_{x}\, (0.5)^{x} (0.5)^{10-x}$

$\boxed{p(x) = {}^{10}c_{x}\, (0.5)^{10}}$

$p(x \geq 6) = p(x=6) + p(x=7) + p(x=8) + p(x=9) + p(x=10)$

$p(x \geq 6) = 0.2051 + 0.1172 + 0.0439 + 0.0098 + 0.0010$

$\boxed{p(x \geq 6) = 0.3770}$

$n = 10, \quad p = \frac{1}{4}, \quad q = \frac{3}{4}$

$p(x) = {}^{n}c_{x}\, p^{x}\, q^{n-x}$

$\boxed{p(x \geq 6) = {}^{10}c_{x} \left(\frac{1}{4}\right)^{x} \left(\frac{3}{4}\right)^{10-x}}$

$p(x \geq 6) = p(6) + p(7) + p(8) + p(9) + p(10)$

$p(x \geq 6) = 0.0162 + 0.0031 + 0.0004 + 0 + 0$

$\boxed{p(x \geq 6) = 0.0197}$

5. The number of telephone lines busy at any instant of time is a binomial variate with probability that a line is busy is 0.1. If 10 lines are choosen at random what is probability that

(i) No line is busy
(ii) Atleast one line is busy
(iii) Atmost two lines are busy

sol $p = 0.1 \quad q = 0.9 \quad n = 10$

Binomial distribution :-

$p(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$

$$\boxed{p(x) = {}^{10}C_{x}\,(0.1)^{x}\,(0.9)^{10-x}}$$

(i) $\boxed{p(x=0) = 0.3487}$

(ii) $p(x \geq 1) = 1 - p(x<1)$

$p(x \geq 1) = 1 - p(x=0)$

$p(x \geq 1) = 1 - 0.3487$

$\boxed{p(x \geq 1) = 0.6513}$

(iii) $p(x \leq 2) = p(0) + p(1) + p(2)$

$p(x \leq 2) = 0.3487 + 0.3874 + 0.1937$

$\boxed{p(x \leq 2) = 0.9298}$

*** 6. Jan 2024 When a coin is tossed 4 times find the probability of getting

(i) Exactly 1 head
(ii) Atleast 3 heads
(iii) Less than 2 heads
(iv) Atmost 3 heads

sol $p = \frac{1}{2} \quad q = 1 - \frac{1}{2} = \frac{1}{2} \quad n = 4$

By Binomial distribution

$p(x) = {}^nC_x \; p^x \; q^{n-x}$

$\boxed{p(x) = {}^4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}}$

(i) $\boxed{p(x=1) = 0.25}$

(ii) $p(x \geq 3) = p(3) + p(4)$

$p(x \geq 3) = 0.25 + 0.0625$

$\boxed{p(x \geq 3) = 0.3125}$

(iii) $p(x<2) = p(0) + p(1)$

$p(x<2) = 0.0625 + 0.25$

$\boxed{p(x<2) = 0.3125}$

(iv) $p(x \leq 3) = p(0) + p(1) + p(2) + p(3)$

$p(x \leq 3) = 0.0625 + 0.25 + 0.375 + 0.25$

$\boxed{p(x \leq 3) = 0.9375}$

*** MQP

7.* MQP The probability a pen manufactured by a factory is defective is 0.1, if 12 such pens are manufactured what is probability that

(i) Exactly 2 are defective
(ii) Atleast 2 are defective
(iii) None of them are defective
(iv) Atleast 1 is defective

$p = 0.1 \quad q = 0.9 \quad n = 12$

By Binomial distribution

$p(x) = {}^nC_x \; p^x \; q^{n-x}$

$p(x) = {}^{12}C_x \; (0.1)^x \; (0.9)^{12-x}$

(i) $\boxed{p(x=2) = 0.2301}$

(ii) $p(x \geq 2) = 1 - p(x<2)$

$p(x \geq 2) = 1 - [p(x=0) + p(x=1)]$

$p(x \geq 2) = 1 - 0.2824 - 0.3766$
$p(x \geq 2) = 0.3410$

(iii) $p(x=0) = 0.2824$

(iv) $p(x \geq 1) = 1 - [p(x=0)]$
$p(x \geq 1) = 1 - 0.2824$
$\boxed{p(x \geq 1) = 0.7176}$

8. An airline knows that 5% of the people making reservation on a certain flight will not turn up consequently their policy is to sell 52 tickets for a flight that can only hold 50 people what is the probability that there will be seat for every passenger who turn up?

sol Let 'X' denote number of passengers who do not turn up and let 'p' be probability of passenger who do not turn up

$n = 52 \quad p = 0.05 \quad q = 0.95$

By binomial distribution

$p(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$

$p(x) = {}^{52}C_{x}\, p^{x}\, q^{52-x} \Rightarrow \boxed{p(x) = {}^{52}C_{x}\, (0.05)^{x}\, (0.95)^{52-x}}$

$p(x \geq 2) = 1 - p(x<2)$
$p(x \geq 2) = 1 - [p(x=0) + p(x=1)]$
$p(x \geq 2) = 1 - 0.0694 - 0.1901$
$\boxed{p(x \geq 2) = 0.7405}$

*** MQP
9 MQP In 800 families with 5 children each, how many families would be expected to have
(i) 3 boys
(ii) 5 girls

(iii) either 2 or 3 boys

(iv) Atmost 2 girls by assuming probabilities for boys and girls to be equal

(iv) Atleast 1 boy

(vi) Atmost 2 boys

sol Let 'x' denote number of boys

$n=5, \quad p=\frac{1}{2}, \quad q=\frac{1}{2}$

$p(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

$p(x) = {}^{5}C_{x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{5-x}$

$p(x) = \frac{5C_x}{32}$

Expected number with respect to 800 families

$= 800\,p(x) = \frac{800 \times 5C_x}{32} = \boxed{25\,[5C_x]} = f(x)$

(i) $\boxed{f(x=3) = 250}$

(ii) 5 girls means 0 boy $\Rightarrow$ $\boxed{f(0) = 25}$

(iii) $f(2)+f(3) = 250+250$

$\boxed{f(2)+f(3) = 500}$

(iv) Atmost 2 girls mean 5 boys & 0 girls or 4 boys & 1 girl or 3 boys & 2 girls

$f(5)+f(4)+f(3) = 25+125+250$

$\boxed{f(5)+f(4)+f(3) = 250}$

(v) $f(1)+f(2)+f(3)+f(4)+f(5) = 125+250+250+125+25$

$\boxed{f(1)+f(2)+f(3)+f(4)+f(5) = 775}$

(vi) $f(0)+f(1)+f(2) = 25+125+250$

$\boxed{f(0)+f(1)+f(2) = 400}$

10. 4 coins are tossed 100 times and the following results are obtained. fit a probability distribution for the data and calculate theoretical frequency

No. of heads :	0	1	2	3	4
f :	5	29	36	25	5
fx :	0	29	72	75	20

sol

$$\text{Mean} = \frac{\Sigma fx}{\Sigma f}$$

$$\text{Mean} = \frac{196}{100}$$

$$\text{Mean} = 1.96$$

$$np = 1.96$$

$$4p = 1.96$$

$$p = \frac{1.96}{4}$$

$$\boxed{p = 0.49}$$

$$q = 1 - p$$

$$\boxed{q = 0.51}$$

$$p(x) = {}^{n}c_{x}\, p^{x} q^{n-x}$$

$$\boxed{p(x) = {}^{4}c_{x}\, (0.49)^{x} (0.51)^{4-x}}$$

$$\boxed{F(x) = 100 \times {}^{4}c_{x}\, (0.49)^{x} (0.51)^{4-x}}$$

Theoretical frequencies

$F(0) = 6.7 \simeq 7$

$F(1) = 26.6 \simeq 26$

$F(2) = 37.5 \simeq 37$

$F(3) = 24$

$F(4) = 5.8 \simeq 6$

No of heads	:	0	1	2	3	4
T.F	:	7	26	37	24	6

11. The probability of a shooter hitting a target is $\frac{1}{3}$. How many times he should shoot so that the probability of hitting the target atleast once is more than $\frac{3}{4}$?

sol Let 'X' denote number of times shooter hits the target

$$p(x \geq 1) > \frac{3}{4}$$

$$1 - p(x < 1) > \frac{3}{4}$$

$$1 - p(x = 0) > \frac{3}{4}$$

$$1 - {}^{n}C_{0} \left(\frac{1}{3}\right)^{0} \left(\frac{2}{3}\right)^{n-0} > \frac{3}{4}$$

$$1 - \left(\frac{2}{3}\right)^{n} > \frac{3}{4}$$

$$1 - \frac{3}{4} > \left(\frac{2}{3}\right)^{n}$$

$$\frac{1}{4} > \left(\frac{2}{3}\right)^{n}$$

$$\left(\frac{2}{3}\right)^{n} < \frac{1}{4}$$

By inspection

$\boxed{n=4}$

By inspection

$n=1, \left(\frac{2}{3}\right)^{1} < \frac{1}{4} \Rightarrow 0.67 < 0.25$

$n=2, \left(\frac{2}{3}\right)^{2} < \frac{1}{4} \Rightarrow 0.44 < 0.25$

$n=3, \left(\frac{2}{3}\right)^{3} < \frac{1}{4} \Rightarrow 0.30 < 0.25$

$n=4, \left(\frac{2}{4}\right)^{4} < \frac{1}{4} \Rightarrow 0.20 < 0.25$

12. The number of accidents in a year to taxi drivers in a city follows poisson distribution with mean 3. Out of 1000 taxi drivers find approximately the number of drivers with
(i) With no accident in a year
(ii) More than 3 accidents in a year

sol mean = 3

$m = 3$

$$p(x) = \frac{m^x e^{-m}}{x!}$$

$$\boxed{p(x) = \frac{3^x e^{-3}}{x!}}$$

(i) $p(x=0) = 0.0498$

number of drivers with no accident in a year

$= 0.0498 \times 1000$

$= 49.8$

$\boxed{\simeq 50 \text{ drivers}}$

(ii) $p(x>3) = 1 - p(x \leq 3)$

$p(x>3) = 1 - [p(x=0) + p(x=1) + p(x=2) + p(x=3)]$

$p(x>3) = 1 - [0.0498 + 0.1494 + 0.2240 + 0.2240]$

$\boxed{p(x>3) = 0.3528}$

number of drivers $= 1000 \times 0.3528$

number of drivers $= 352.8$

$\boxed{\text{number of drivers} = 353 \text{ drivers}}$

*** MQP

13. If a probability of bad reaction in a certain injection is 0.001. Determine the chance out of 2000 individuals more than 2 will get a bad reaction, also find the chance that exactly 3 will suffer from bad reaction

MQP

sol $n = 2000$ $p = 0.001$

mean $(m) = np$

mean $(m) = 2000 \times 0.001$

mean $(m) = 2$

By poisson distribution

$p(x) = \dfrac{m^x e^{-m}}{x!}$

$$\boxed{p(x) = \frac{2^x e^{-2}}{x!}}$$

(i) $p(x>2) = 1 - [p(x \le 2)]$

$p(x>2) = 1 - [p(0) + p(1) + p(2)]$

$p(x>2) = 1 - [0.1353 + 0.2707 + 0.2707]$

$\boxed{p(x>2) = 0.3233}$

(ii) $\boxed{p(x=3) = 0.1804}$

*** MQP

14. In a certain factory turning out razor blades there is a small probability of $\frac{1}{500}$ for any blade to be defective. the blades are supplied in packets of 10. find the number of packets containing

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(i) One defective blade

(ii) 2 defective blade

(iii) no defective blade

In a consignment of 10,000 packets

$p = 0.002$ $n = 10$

$m = np = 0.002 \times 10$

$m = 0.02$

By poisson distribution

$$\boxed{p(x) = \frac{m^x e^{-m}}{x!}}$$

$$p(x) = \frac{(0.02)^x e^{-0.02}}{x!} = \boxed{\frac{10{,}000 \times (0.02)^x e^{-0.02}}{x!}} = f(x)$$

(i) $f(x=1) = 196.0397 \simeq \boxed{196 \text{ packets}}$

Number of packets with 1 defective blade 196 packets

(ii) $f(x=2) = 1.9603 \simeq \boxed{2 \text{ packets}}$

Number of packets with 2 defective blades is 2 packets

(iii) $f(x=0) = 9801.9867 \simeq \boxed{9802 \text{ packets}}$

Number of packets with no defective blades is 9802 packets

15. Alpha particles are emitted by a radioactive source at an average rate of 5 in 20 minutes interval using poisson distribution find the probability that there will be

(i) Exactly 2 emission

(ii) Atleast 2 emission in randomly choosen 20 minut

sol $m = 5$

By poisson distribution

$$p(x) = \frac{m^x e^{-m}}{x!}$$

$$p(x) = \frac{5^x e^{-5}}{x!}$$

(i) $\boxed{p(x=2) = 0.0842}$

(ii) $p(x \geq 2) = 1 - [p(x<2)]$

$p(x \geq 2) = 1 - [p(0) + p(1)]$

$p(x \geq 2) = 1 - [0.0067 + 0.0337]$

$\boxed{p(x \geq 2) = 0.9596}$

16. Fit a poisson distribution to the following set of observation

x	0	1	2	3	4
$f(x)$	122	60	15	2	1
fx	0	60	30	6	4

sol

$$\text{Mean} = \frac{\Sigma xfx}{\Sigma f} \Rightarrow \frac{0+60+30+6+4}{122+60+15+2+1}$$

$$\text{Mean} = \frac{100}{200}$$

$\boxed{\text{Mean} = 0.5}$

By poisson distribution

$$\boxed{p(x) = \frac{m^x e^{-m}}{x!} = \frac{(0.5)^x e^{-0.5}}{x!}}$$

$f(0) = 121.30 \simeq 121$

$f(1) = 60.66 \simeq 61$

$f(2) = 15.16 \simeq 15$

$f(3) = 2.52 \simeq 3$

$f(4) = 0.32 \simeq 0$

No. of observations = $122+60+15+2+1 = 200$

$$\boxed{f(x) = 200 \times p(x) = \frac{200 \times (0.5)^x e^{-0.5}}{x!}}$$

poisson distribution

x :	0	1	2	3	4
$f(x)$:	121	61	15	3	0

17. If x is a poisson variate such that $p(x=2) = 9\,p(x=4) + 90\,p(x=6)$ compute mean and variance of poisson's distribution

sol consider poisson distribution

$$p(x) = \frac{m^x e^{-m}}{x!}$$

$$p(x=2) = 9\,p(x=4) + 90\,p(x=6)$$

$$\frac{m^2 e^{-m}}{2!} = \frac{9 m^4 e^{-m}}{4!} + \frac{90 m^6 e^{-m}}{6!}$$

$\div$ by $m^2 e^{-m}$

$$\frac{1}{2} = \frac{9m^2}{4\times3\times2\times1} + \frac{90m^4}{6\times5\times4\times3\times2\times1}$$

$$\frac{1}{2} = \frac{3m^2}{8} + \frac{m^4}{8}$$

$\times^{ly}$ by 8

$$4 = 3m^2 + m^4$$

$$m^4 + 3m^2 - 4 = 0$$

$$m^4 + 4m^2 - 1m^2 - 4 = 0$$

$$m^2(m^2+4) - 1(m^2+4) = 0$$

$m^2 - 1 = 0$ and $m^2 + 4 = 0$

$m = \pm 1$ $\quad$ $m = \pm 2i$

Neglecting negative and imaginary values

$$\boxed{\begin{array}{l} m = 1 \\ \text{mean} = \text{variance} = 1 \end{array}}$$

⇒ **Continuous probability distribution**

p.d.f and c.d.f (probability density function and cumulative density function)

$$\boxed{\begin{array}{l} f(x) \text{ is said to be p.d.f if} \\ \text{(i) } f(x) \geq 0 \\ \text{(ii) } \int_{-\infty}^{\infty} f(x)\,dx = 1 \end{array}}$$

if x is a continuous random variable with p.d.f $f(x)$ then c.d.f of x is given by

$$\boxed{F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx}$$

⇒ **Mean and variance**

$$\text{Mean } (\mu) = \int_{-\infty}^{\infty} x f(x)\,dx = E(x)$$

$$\text{Variance } (\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} x f(x)\,dx\right]^2$$

$$\boxed{\text{Variance } (\sigma^2) = E(x^2) - [E(x)]^2}$$

⇒ **Exponential distribution**

The continuous probability distribution having p.d.f given by

$$\boxed{f(x) = \begin{cases} \alpha e^{-\alpha x} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}}$$

where α is positive is known as exponential distribution

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0 + \int_{0}^{\infty} \alpha e^{-\alpha x}\,dx = \alpha \left[\frac{e^{-\alpha x}}{-\alpha}\right]_0^{\infty} = -\left[e^{-\alpha x}\right]_0^{\infty}$

$= -[0-1] = 1$

Note: for exponential distribution

$\boxed{\mu = \frac{1}{\alpha}}$, $\boxed{V = \frac{1}{\alpha^2}}$, $\boxed{\sigma = \frac{1}{\alpha}}$

2) **Normal distribution**

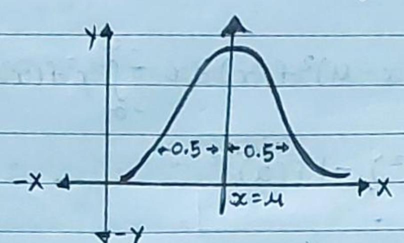
The continuous probability distribution having p.d.f $\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}}$ where, $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$

is known as normal distribution

(i) **Note**: for normal distribution

$\boxed{Mean = \mu}$ $\boxed{V = \sigma^2}$ $\boxed{S.D = \sigma}$

(ii) The graph of probability function $f(x)$ is a bell shaped curve symmetrical about the line $x = \mu$ is called normal probability curve



The line $x = \mu$ divides the total area under the curve which is equal to 1 into 2 equal parts the area to right as to the left of the line $(x = \mu)$ is 0.5

(iii) $z = \frac{x - \mu}{\sigma}$ is called standard normal variate

⇒ Problems

1. Find which of the following is a p.d.f

(i) $f_1(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

(ii) $f_2(x) = \begin{cases} 2x, & -1 < x < 0 \\ 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

sol (i) a) $f_1(x) \geq 0$ is true

b) $\int_{-\infty}^{\infty} f_1(x)\,dx = \int_{-\infty}^{0} 0 + \int_{0}^{1} 2x\,dx + \int_{1}^{\infty} 0$

$\int_{-\infty}^{\infty} f_1(x)\,dx = 2\frac{x^2}{2}\Big]_0^1$

$\int_{-\infty}^{\infty} f_1(x)\,dx = [1-0]$

$\boxed{\int_{-\infty}^{\infty} f_1(x)\,dx = 1}$

$f_1(x)$ is p.d.f

(ii) For $-1 < x < 0$, $f_2(x) < 0$

$\therefore f_2(x) \ngeq 0$

$\therefore f_2(x)$ is not a p.d.f

2. Find the value of K such that the function

$f(x) = \begin{cases} Kx^2, & 1 < x < 3 \\ 0, & \text{otherwise} \end{cases}$ is a p.d.f of a continuous random variable. also find $P(1.5 < x < 2.5)$

sol (i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{1} 0 + \int_{1}^{3} Kx^2 dx + \int_{3}^{\infty} 0 = 1$$

$$K \frac{x^3}{3}\Big]_1^3 = 1$$

$$\frac{K}{3}[27-1] = 1$$

$$\frac{26K}{3} = 1$$

$$\boxed{K = \frac{3}{26}}$$

$$f(x) = \begin{cases} \frac{3x^2}{26}, & 1 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

$$P(1.5 < x < 2.5) = \int_{1.5}^{2.5} f(x)dx$$

$$P(1.5 < x < 2.5) = \frac{3}{26}\int_{1.5}^{2.5} x^2 dx$$

$$P(1.5 < x < 2.5) = \frac{\cancel{3}}{26} \frac{x^3}{\cancel{3}}\Big]_{1.5}^{2.5}$$

$$P(1.5 < x < 2.5) = \frac{1}{26}[(2.5)^3 - (1.5)^3]$$

$$\boxed{P(1.5 < x < 2.5) = 0.4712}$$

** Jan 2025

* 3. A random variable x has density function.

Dec 2023

$$p(x) = \begin{cases} Kx^2, & -3 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$ evaluate K and find

(i) $P(1 \le x \le 2)$

(ii) $P(x \le 2)$

(iii) $P(x > 1)$

sol (i) $p(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} p(x)\,dx = 1$

$$\int_{-\infty}^{-3} 0 + \int_{-3}^{3} p(x)\,dx + \int_{3}^{\infty} 0 = 1$$

$$\int_{-3}^{3} \frac{1}{18} x^2\,dx = 1$$

$$\frac{1}{18}\frac{x^3}{3}\Big]_{-3}^{3} = 1$$

$$\frac{1}{18\times 3}[27+27] = 1$$

$$p(x) = \begin{cases} \frac{x^2}{18}, & -3 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

$$18K = 1$$

$$\boxed{K = \frac{1}{18}}$$

(i) $P(1 \leq x \leq 2) = \int_{1}^{2} \frac{x^2}{18}\,dx$

$P(1 \leq x \leq 2) = \frac{1}{18}\frac{x^3}{3}\Big]_{1}^{2}$

$P(1 \leq x \leq 2) = \frac{1}{54}[8-1]$

$P(1 \leq x \leq 2) = \frac{7}{54} \Rightarrow \boxed{P(1 \leq x \leq 2) = 0.1296}$

(ii) $P(x \leq 2) = \int_{-3}^{2} \frac{x^2}{18}\,dx$

$P(x \leq 2) = \frac{1}{18}\frac{x^3}{3}\Big]_{-3}^{2}$

$P(x \leq 2) = \frac{1}{18\times 3}[8+27]$

$P(x \leq 2) = \frac{1}{54}[35]$

$P(x \leq 2) = \frac{35}{54}$

$$\boxed{P(x \leq 2) = 0.6481}$$

(iii) $P(x>1) = \int_{1}^{3} \frac{x^2}{18} dx$

$P(x>1) = \frac{1}{18} \left. \frac{x^3}{3} \right]_1^3$

$P(x>1) = \frac{1}{54} [27-1]$

$P(x>1) = \frac{26}{54}$

$P(x>1) = \frac{13}{27} \Rightarrow \boxed{P(x>1) = 0.4815}$

4. Find the constant K such that

$$f(x) = \begin{cases} Kx^2, & 0<x<3 \\ 0, & \text{otherwise} \end{cases} \text{ is a p.d.f.}$$

Find
(i) $P(1 \leq x \leq 2)$
(ii) $P(x>1)$
(iii) Mean
(iv) Variance

sol (i) $f(x) \geq 0$
(ii) $\int_{-\infty}^{\infty} f(x) dx = 1$

$\int_{-\infty}^{0} 0 + \int_{0}^{3} f(x) dx + \int_{3}^{\infty} 0 = 1$

$\int_{0}^{3} Kx^2 = 1$

$$f(x) = \begin{cases} \frac{x^2}{9}, & 0<x<3 \\ 0, & \text{otherwise} \end{cases}$$

$K \left. \frac{x^3}{3} \right]_0^3 = 1$

$\frac{K}{3} [27-0] = 1$

$9K = 1$

$\boxed{K = \frac{1}{9}}$

(i) $P(1 \le x \le 2) = \int_1^2 f(x)\,dx$

$P(1 \le x \le 2) = \int_1^2 \frac{x^2}{9}\,dx$

$P(1 \le x \le 2) = \frac{1}{9} \left.\frac{x^3}{3}\right]_1^2$

$P(1 \le x \le 2) = \frac{1}{27} \{[8-1]\}$

$P(1 \le x \le 2) = \frac{7}{27}$

$\boxed{P(1 \le x \le 2) = 0.2592}$

(ii) $P(x > 1) = \int_1^3 f(x)\,dx$

$P(x > 1) = \int_1^3 \frac{x^2}{9}\,dx$

$P(x > 1) = \frac{1}{9} \left.\frac{x^3}{3}\right]_1^3$

$P(x > 1) = \frac{1}{9 \times 3} \{[27-1]\}$

$P(x > 1) = \frac{26}{27}$

$\boxed{P(x > 1) = 0.9630}$

(iii) $\text{Mean}(\mu) = \int_0^3 x f(x)\,dx$

$\text{Mean}(\mu) = \int_0^3 x \frac{x^2}{9}$

$\text{Mean}(\mu) = \frac{1}{9}\left\{\left.\frac{x^4}{4}\right]_0^3\right\}$

$\text{Mean}(\mu) = \frac{1}{36}[81-0]$

$\text{Mean}(\mu) = \frac{81}{36} \Rightarrow \boxed{\text{Mean}(\mu) = 2.25}$

iv) $\text{Variance} = \int_{-\infty}^{\infty} x^2 f(x)\,dx - (\mu)^2$

$\text{Variance} = \frac{1}{9}\int_0^3 x^2\, x^2\, dx - (2.25)^2$

$\text{Variance} = \frac{1}{9}\int_0^3 x^4\, dx - (2.25)^2$

$\text{Variance} = \frac{1}{9}\left.\frac{x^5}{5}\right]_0^3 - (2.25)^2$

$\text{Variance} = \frac{243}{45} - (2.25)^2$

$\text{Variance} = 5.4 - 5.0625$

$\boxed{\text{Variance} = 0.3375}$

5. Find the c.d.f for the p.d.f

(i) $f(x) = \begin{cases} 6x - 6x^2, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

(ii) $f(x) = \begin{cases} x/4\, e^{-x/2}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$

(iii) Exponential distribution

sol $\text{c.d.f} = F(x) = \int_{-\infty}^{x} f(x)\,dx$

$\text{c.d.f} = F(x) = \int_{-\infty}^{0} 0 + \int_0^x (6x - 6x^2)\,dx \quad ; \quad 0 \le x \le 1$

$\text{c.d.f} = F(x) = \frac{{}^{3}\cancel{6}x^2}{\cancel{2}_1} - \frac{{}^{2}\cancel{6}x^3}{\cancel{3}_1}$

$\boxed{\text{c.d.f} = F(x) = 3x^2 - 2x^3}$

$$c.d.f = F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$c.d.f = F(x) = \int_{-\infty}^{0} 0 + \int_{0}^{x} f(x)\,dx$$

$$c.d.f = F(x) = \int_{0}^{x} \frac{x}{4} e^{-x/2}$$

$$c.d.f = F(x) = \frac{1}{4}\int_{0}^{x} x e^{-x/2}\,dx$$

$$c.d.f = F(x) = \frac{1}{4}\left\{ x\frac{e^{-x/2}}{-1/2} - \int (1)\frac{e^{-x/2}}{-1/2} \right\}\Big]_{0}^{x}$$

$$c.d.f = F(x) = \frac{1}{4}\left\{ x\frac{e^{-x/2}}{-1/2} - \frac{e^{-x/2}}{-1/2 \times -1/2} \right\}\Big]_{0}^{x}$$

$$c.d.f = F(x) = \frac{1}{4}\left\{ -2xe^{-x/2} - 4e^{-x/2} \Big]_{0}^{x} \right\}$$

$$c.d.f = F(x) = \frac{1}{4}\left\{ -2xe^{-x/2} - 4e^{-x/2} + 4 \right\}$$

$$c.d.f = F(x) = -\frac{xe^{-x/2}}{2} - e^{-x/2} + 1$$

$$\boxed{c.d.f = F(x) = 1 - e^{-x/2} - \frac{xe^{-x/2}}{2}}$$

(iii) $$f(x) = \begin{cases} \alpha e^{-\alpha x}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

$$c.d.f = F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$c.d.f = F(x) = \int_{-\infty}^{0} 0 + \int_{0}^{x} f(x)\,dx$$

$$c.d.f = F(x) = \int_{0}^{x} \alpha e^{-\alpha x}\,dx$$

$$c.d.f = F(x) = \alpha \frac{e^{-\alpha x}}{-\alpha}\Big]_{0}^{x}$$

$$c.d.f = F(x) = -[e^{-\alpha x} - e^{0}]$$

$$\boxed{c.d.f = F(x) = [1 - e^{-\alpha x}}$$

* 6. Density function of random variable is given by
MQP

$$f(x) = \begin{cases} K\sqrt{x}, & 0<x<1 \\ 0, & \text{elsewhere} \end{cases}$$

(i) Find K

(ii) Find c.d.f and use it to evaluate $P(0.3<x<0.6)$

(i) c.d.f $= F(x) = \int_{-\infty}^{x} f(x)\,dx = 1$

$$\int_{-\infty}^{0} 0 + \int_{0}^{1} f(x)\,dx = 1$$

$$\int_{0}^{1} K x^{1/2}\,dx = 1$$

$$K \left.\frac{x^{3/2}}{3/2}\right]_0^1 = 1$$

$$\frac{2K}{3}[1-0] = 1$$

$$\boxed{K = \frac{3}{2}}$$

$$f(x) = \begin{cases} \frac{3\sqrt{x}}{2}, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$

(ii) c.d.f $= F(x) = \int_{-\infty}^{x} f(x)\,dx$

c.d.f $= F(x) = \int_{-\infty}^{0} 0 + \int_{0}^{x} f(x)\,dx$

c.d.f $= F(x) = \int_{0}^{x} \frac{3x^{1/2}}{2}$

c.d.f $= F(x) = \left.\frac{3}{2} \times \frac{x^{3/2}}{3/2}\right]_0^x$

c.d.f $= F(x) = \left.\cancel{\frac{3}{2}} \times \cancel{\frac{2}{3}}\, x^{3/2}\right]_0^x$

$$\boxed{\text{c.d.f} = F(x) = x^{3/2}}$$

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$F(0.3<x<0.6) = P(0<x<0.6) - P(0<x<0.3)$

$F(0.3<x<0.6) = F(0.6) - F(0.3)$

$F(0.3<x<0.6) = 0.4648 - 0.1643$

$\boxed{F(0.3<x<0.6) = 0.3005}$

7. Evaluate using normal distribution table the followi

(i) $P(z \geq 0.85)$

(ii) $P(-1.64 \leq z \leq -0.88)$

(iii) $P(z \leq -2.43)$

(iv) $P(|z| \leq 1.94)$

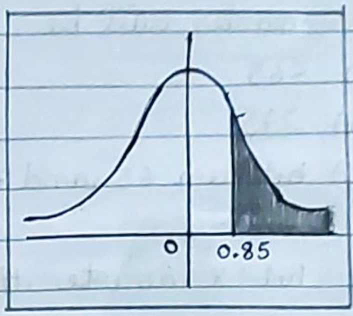
(i) $P(z \geq 0.85)$

sol $= 0.5 - P(0 \leq z \leq 0.85)$

$= 0.5 - \phi(0.85)$

$= 0.5 - 0.3023$

$\boxed{= 0.1977}$



(ii) $P(-1.64 \leq z \leq -0.88)$

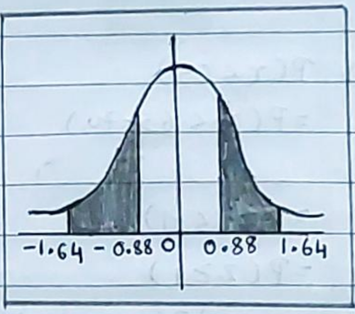
sol $= P(0.88 \leq z \leq 1.64)$

$= P(0 \leq z \leq 1.64) - P(0 \leq z \leq 0.88)$

$= \phi(1.64) - \phi(0.88)$

$= 0.4495 - 0.3106$

$\boxed{= 0.1389}$



(iii) $P(z \leq -2.43)$

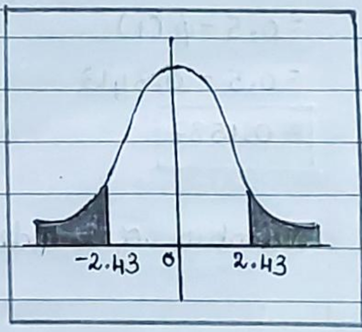
sol $= P(z \geq 2.43)$

$= 0.5 - P(0 \leq z \leq 2.43)$

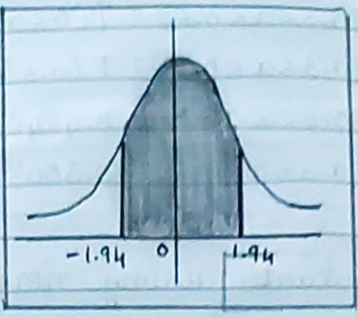
$= 0.5 - \phi(2.43)$

$= 0.5 - 0.4925$

$\boxed{= 0.0075}$



(iv) $P(|z| \leq 1.94)$
$= P(-1.94 \leq z \leq 1.94)$
$= 2P(0 \leq z \leq 1.94)$
$= 2\phi(1.94)$
$= 2 \times 0.4738$
$= 0.9476$



** Jan 2025

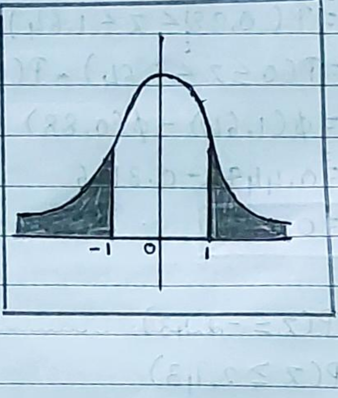
8. The marks of 1000 students in an exam follows normal distribution with mean $(\mu) = 70$ and standard deviation $(\sigma) = 5$. Find the number of students whose marks will be
*

(i) < 65
(ii) > 75
(iii) between 65 and 75

sol Let 'X' denote the marks of the students
$\mu = 70$ $\sigma = 5$ $z = \frac{x-\mu}{\sigma} = \frac{x-70}{5}$

(i) $P(x < 65)$
$= P(z < \frac{65-70}{5})$

$= P(z < -1)$
$= P(z \geq 1)$
$= 0.5 - P(0 < z < 1)$
$= 0.5 - \phi(1)$
$= 0.5 - 0.3413$
$\boxed{= 0.1587}$



Number of students having marks < 65
$= 0.1587 \times 1000$
$= 158.7$
$\boxed{\simeq 159 \text{ students}}$

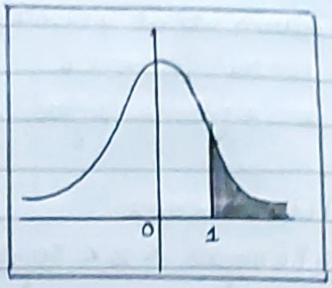
(ii) $P(x > 75)$

$= P\left(z > \frac{75-70}{5}\right)$

$= P(z > 1)$

$= 0.5 - P(0 < z < 1)$

$\boxed{= 0.1587}$



Number of students having marks >75

$= 0.1587 \times 1000$

$= 158.7$

$\boxed{\simeq 159 \text{ students}}$

(iii) $P(65 < x < 75)$

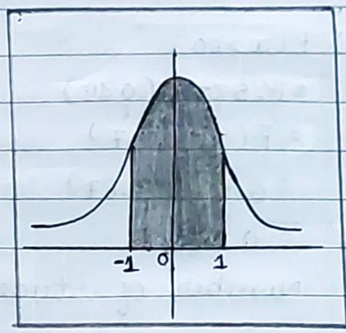
$= P\left(\frac{65-70}{5} < z < \frac{75-70}{5}\right)$

$= P(-1 < z < 1)$

$= 2P(0 < z < 1)$

$= 2 \times 0.3413$

$\boxed{= 0.6826}$



Number of students scoring marks between 65 and 75

$= 0.6826 \times 1000$

$= 682.6$

$\boxed{\simeq 683 \text{ students}}$

9.* The mean weight of 500 students during a medical exam was found to be 50 kg and standard deviation weight 6 kg. Assuming that the weights are normally distributed. Find the number of students having weight

(i) Between 40 and 50 kg

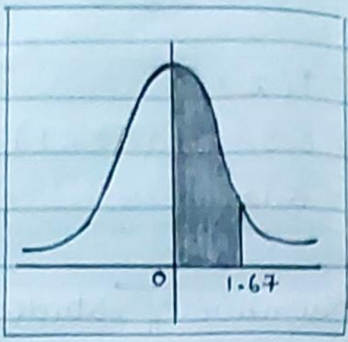
(ii) More than 60 kg

Given that $\phi(1.67) = 0.4525$

sol Given that $\phi(1.67) = 0.4525$

$\mu = 50 \quad \sigma = 6$

$z = \frac{x-\mu}{\sigma} = \frac{x-50}{6}$



(i) $P(40 < x < 50)$

$= P\left(\frac{40-50}{6} < z < \frac{50-50}{6}\right)$

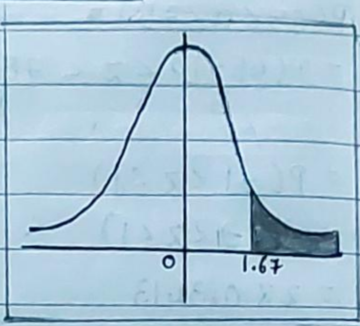
$= P(-1.67 < z < 0)$

$= \phi(1.67)$

$= 0.4525$

Number of students $= 500 \times 0.4525 = 226.25$

$\boxed{\simeq 226 \text{ students}}$



(ii) $P(x > 60)$

$= P\left(z > \frac{60-30}{6}\right)$

$= P(z > 1.67)$

$= 0.5 - \phi(1.67)$

$= 0.0475$

Number of students $= 500 \times 0.0475 = 23.75$

$\boxed{\simeq 24 \text{ students}}$

*** 10. In an examination 7% of the students score less than 35% marks and 89% of students score less than 60% marks. Find the mean and standard deviation if the marks are normally distributed. It is given that

June 2012 7M

$P(0 < Z < 1.2263) = 0.39$, $P(0 < z < 1.4757) = 0.43$

sol

$\phi(1.4757) = 0.43$

$P(x < 35) = \frac{7}{100} = 0.07$, $P(x < 60) = 0.89$

$z = \frac{x-\mu}{\sigma}$

when $x = 35$, $z_1 = \frac{35-\mu}{\sigma}$

when $x = 60$, $z_2 = \frac{60-\mu}{\sigma}$

$P(x<35) = 0.07$

$P(z<z_1) = 0.07$

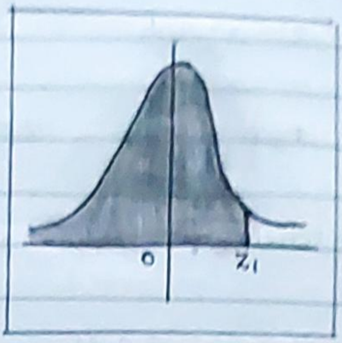
$0.5 + P(0<z<z_1) = 0.07$

$0.5 + \phi(z_1) = 0.07$

$\phi(z_1) = -0.43$

$\phi(z_1) = -\phi(1.4757)$

$\boxed{z_1 = -1.4757}$



$P(x<60) = 0.89$

$P(z<z_2) = 0.89$

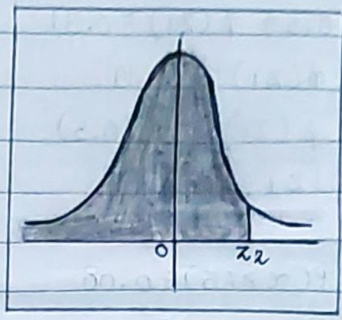
$0.5 + P(0<z<z_2) = 0.89$

$0.5 + \phi(z_2) = 0.89$

$\phi(z_2) = 0.39$

$\phi(z_2) = \phi(1.2263)$

$\boxed{z_2 = 1.2263}$



$-1.4757\sigma = 35 - \mu$

$1.2263\sigma = 60 - \mu$

$\mu - 1.4757\sigma = 35$

$\mu + 1.2263\sigma = 60$

solving using calculator

$\boxed{\mu = 48.65}$

$\boxed{\sigma = 9.25}$

** MQP

* 11. In a normal distribution 31% of items are under 45 and 8% are over 64, find mean and standard deviation given that $A(0.5) = 0.19$ and $A(1.4) = 0.42$ where $A(z)$ is area under standard normal curve from 0 to z. Also find the variance

Dec2012 (7M) (MQP)

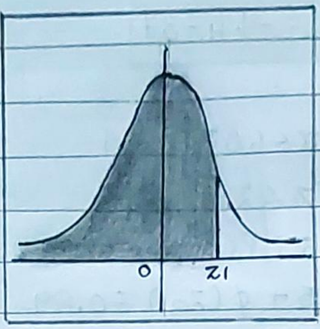
sol $A(0.5) = 0.19$ $\quad A(1.4) = 0.42$

$P(x<45) = \frac{31}{100} = 0.31$, $P(x>64) = \frac{8}{100} = 0.08$

$z = \frac{x-\mu}{\sigma}$

when $x = 45$, $z_1 = \frac{45-\mu}{\sigma}$

when $x = 65$, $z_2 = \frac{64-\mu}{\sigma}$



$P(x<45) = 0.31$

$P(z<z_1) = 0.31$

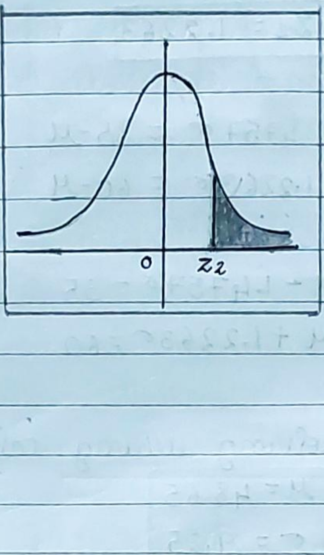
$0.5 + P(0<z<z_1) = 0.31$

$0.5 + \phi(z_1) = 0.31$

$\phi(z_1) = -0.19$

$\phi(z_1) = -\phi(0.5)$

$\boxed{z_1 = -0.5}$



$P(x>65) = 0.08$

$P(z>z_2) = 0.08$

$0.5 - P(0<z<z_2) = 0.08$

$0.5 - \phi(z_2) = 0.08$

$\phi(z_2) = 0.42$

$\phi(z_2) = \phi(0.42)$

$\boxed{z_2 = 1.4}$

$z_1 = \frac{45-\mu}{\sigma}$ $\quad$ $z_2 = \frac{64-\mu}{\sigma}$

$-0.5\sigma = 45-\mu$

$1.4\sigma = 64-\mu$

By calculator $\quad$ $\mu - 0.5\sigma = 45$

$\mu + 1.4\sigma = 64$

$\boxed{\begin{array}{l}\mu = 50 \\ \sigma = 10\end{array}}$

$\boxed{\text{variance} = (\sigma)^2 = 10^2 = 100}$

classmate
Date 28/9/2025
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** Dec 2023

* 12. Dec 2023 7M The life of an electric bulb is normally distributed with average life of 2000 hours and standard deviation of 60 hrs. out of 2500 bulbs find the number of bulbs that are likely to last

i) between 1900 & 2100 hrs

ii) More than 2100 hrs

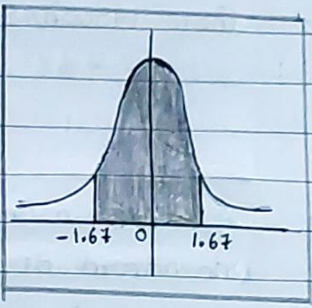
iii) Less than 1950 hrs

[Given $P(0 \le z \le 1.67) = 0.4525$ and $\phi(0.83) = 0.2967$]

sol $\sigma = 60 \quad \mu = 2000$

(i) $P(1900 < x < 2100)$

$= P\left(\frac{1900-2000}{60} < z < \frac{2100-2000}{60}\right)$



$= P(-1.67 < z < 1.67)$

$= 2 \times P(0 < z < 1.67)$

$= 2 \times \phi(1.67)$

$= 2 \times 0.4525$

$\boxed{= 0.905}$

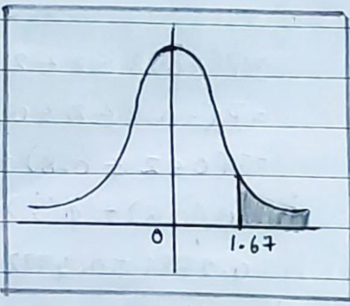
The number of bulbs that are likely to last between 1900 & 2100 hrs $= 0.905 \times 2500$

$= 2262.5$

$\boxed{\simeq 2263 \text{ bulbs}}$

(ii) $P(x > 2100)$

$= P\left(z > \frac{2100-2000}{60}\right)$



$= P(z > 1.67)$

$= 0.5 - P(0 < z < 1.67)$

$= 0.5 - \phi(1.67)$

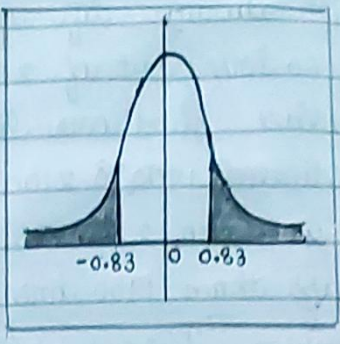
$= 0.5 - 0.4525$

$= 0.0475$

The number of bulbs that are likely to last more than 2100 hrs $= 0.0475 \times 2500 = 118.75$ $\boxed{\simeq 119 \text{ bulbs}}$

(iii) $P(x < 1950)$

$= P\left(z < \frac{1950-2000}{60}\right)$



$= P(z < -0.83)$

$= P(z > 0.83)$

$= 0.5 - P(0 < z < 0.83)$

$= 0.5 - \phi(0.83)$

$= 0.5 - 0.2967$

$\boxed{= 0.2033}$

The number of bulbs that are likely to last less then 1950 hrs $= 0.2033 \times 2500$

$= 508.25$

$\boxed{\approx 508 \text{ bulbs}}$

---

13. If x is a normal variate with mean (μ) 30 and standard deviation (σ) 5 find the probability that

(i) $26 \leq x \leq 40$

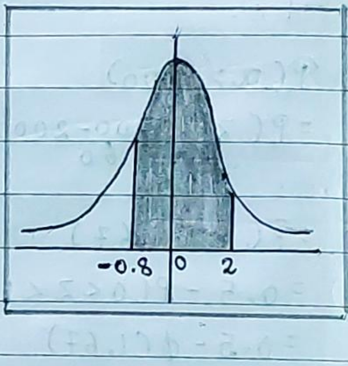
(ii) $x \geq 45$

given that $\phi(0.8) = 0.2881$, $\phi(2) = 0.4772$, $\phi(3) = 0.4987$

$\phi(1) = 0.34$

(i) $P(26 \leq x \leq 40)$ $\qquad z = \frac{x-\mu}{\sigma} = \frac{x-30}{5}$

$= P\left(\frac{26-30}{5} \leq z \leq \frac{40-30}{5}\right)$



$= P(-0.8 \leq z \leq 2)$

$= P(-0.8 \leq z \leq 0) + P(0 \leq z \leq 2)$

$= P(0 \leq z \leq 0.8) + P(0 \leq z \leq 2)$

$= \phi(0.8) + \phi(2)$

$= 0.2881 + 0.4772$

$\boxed{= 0.7653}$

(ii) $P(x \geq 45)$

$P\left(z \geq \frac{45-30}{5}\right)$

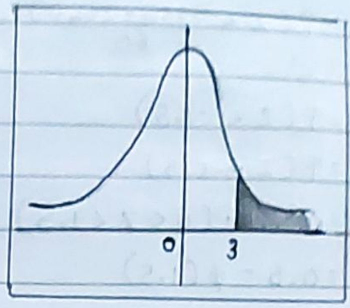
$= P(z \geq 3)$

$= 0.5 - (0 \leq z \leq 3)$

$= 0.5 - \phi(3)$

$= 0.5 - 0.4987$

$\boxed{= 0.0013}$



* ** MQP

14. In a test on 2000 electric bulbs it was found that the life of particular make was normally distributed with an average life 2040 hours and standard deviation 60 hours, estimate the number of bulbs likely to burn

MQP

(i) More than 2150 hours

(ii) Less than 1950 hours

(iii) In between 1920 and 2160 hours

sol (i) $P(x > 2150)$

$z = \frac{x-\mu}{\sigma} = \frac{x-2040}{60}$

$P\left(z > \frac{2150-2040}{60}\right)$

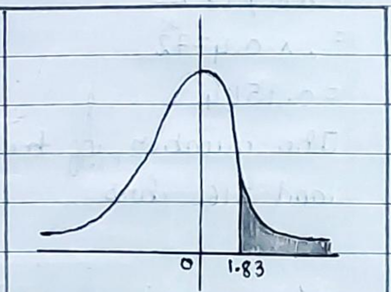
$P(z > 1.83)$

$= 0.5 - P(0 < z < 1.83)$

$= 0.5 - \phi(1.83)$

$= 0.5 - 0.4664$

$= 0.0336$



The number of bulbs likely to burn more than 2150 hrs

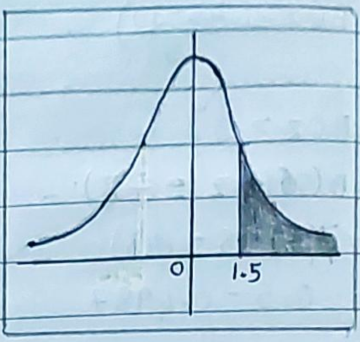
$= 0.0336 \times 2000$

$= 67.2$

$\boxed{= 67 \text{ bulbs}}$

(ii) $P(x < 1950)$

sol $P\left(z < \frac{1950 - 2040}{60}\right)$



$= P(z < -1.5)$

$= P(z > 1.5)$

$= 0.5 - P(0 < z < 1.5)$

$= 0.5 - \phi(1.5)$

$= 0.5 - 0.4332$

$= 0.0668$

The number of bulbs likely to burn less than 1950 hrs

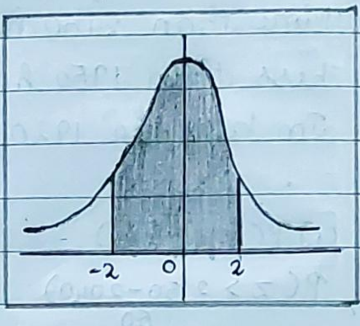
$= 0.0668 \times 2000$

$= 133.6$

$\boxed{= 134 \text{ bulbs}}$

(iii) $P(1920 < x < 2160)$

sol $P\left(\frac{1920 - 2040}{60} < z < \frac{2160 - 2040}{60}\right)$



$= P(-2 < z < 2)$

$= 2P(0 < z < 2)$

$= 2 \times \phi(2)$

$= 2 \times 0.4772$

$= 0.9544$

The number of bulbs likely to burn in between 1920 and 2160 hrs

$= 0.9544 \times 2000$

$= 1908.8$

$\boxed{= 1909 \text{ bulbs}}$

** MQP | Jan 2025

* 7M

15. In a certain town the duration of shower is

June 2013 exponentially distributed having mean 5 minutes

Dec 2023 what is the probability that shower will last

MQP (i) 10 min or more

(ii) < 10 mins

(iii) between 10 and 12 minutes

sol

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{Mean} = \frac{1}{\alpha}$$

$$5 = \frac{1}{\alpha} \Rightarrow \boxed{\alpha = \frac{1}{5}}$$

$$f(x) = \begin{cases} \frac{1}{5} e^{-x/5}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

(i) $P(x \geq 10)$

$$= \frac{1}{5} \int_{10}^{\infty} e^{-x/5} \, dx$$

$$= \frac{1}{5} \left. \frac{e^{-x/5}}{-1/5} \right]_{10}^{\infty}$$

$$= -[0 - e^{-2}]$$

$$= e^{-2}$$

$$\boxed{P(x \geq 10) = 0.1353}$$

(ii) $P(x < 10)$

$$= \frac{1}{5} \int_{0}^{10} e^{-x/5} \, dx$$

$$= \frac{1}{5} \left. \frac{e^{-x/5}}{-1/5} \right]_{0}^{10}$$

$$= -[e^{-2} - 1]$$

$$= -e^{-2} + 1$$

$$\boxed{P(x < 10) = 0.8647}$$

(iii) $P(10 < x < 12)$

$= \frac{1}{5} \int_{10}^{12} e^{-x/5} \, dx$

$= \frac{1}{5} \left. \frac{e^{-x/5}}{-1/5} \right]_{10}^{12}$

$= -[e^{-2.4} - e^{-2}]$

$= -e^{-2.4} + e^{-2}$

$\boxed{P(10 < x < 12) = 0.0446}$

16. The length of telephone conversation in a booth has been an exponential distribution and found on an average to be 5 mins. Find the probability that a random call made from its booth
(i) ends less than 5 min
(ii) ends between 5 and 10 mins

sol $f(x) = \begin{cases} \alpha e^{-\alpha x} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$

Mean $= \frac{1}{\alpha}$

$5 = \frac{1}{\alpha}$

$\boxed{\alpha = \frac{1}{5}}$

$f(x) = \begin{cases} \frac{1}{5} e^{-x/5} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$

(i) $P(x < 5) = \frac{1}{5} \int_{0}^{5} e^{-x/5} \, dx$

$= \frac{1}{5} \left. \frac{e^{-x/5}}{-1/5} \right]_{0}^{5}$

$= -[e^{-1} - e^{0}]$

$\boxed{P(x < 5) = 0.6321}$

(ii) $P(5 \le x \le 10) = \int_5^{10} f(x)dx$

$$= \frac{1}{5}\int_5^{10} e^{-x/5}\,dx$$

$$= \frac{1}{\cancel{5}} \left[\frac{e^{-x/5}}{-1/\cancel{5}}\right]_5^{10}$$

$$= -[e^{-2} - e^{-1}]$$

$$\boxed{P(5 \le x \le 10) = 0.2325}$$

17. If X is an exponential variate with mean 3 find
(i) $P(x>1)$
(ii) $P(x<3)$

sol

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & x>0 \\ 0, & \text{otherwise} \end{cases}$$

Mean $= \frac{1}{\alpha}$

$3 = \frac{1}{\alpha}$

$$\boxed{\alpha = \frac{1}{3}}$$

$$f(x) = \begin{cases} \frac{1}{3}e^{-x/3}, & x>0 \\ 0, & \text{otherwise} \end{cases}$$

(i) $P(x>1) = \frac{1}{3}\int_1^{\infty} e^{-x/3}\,dx$

$$= \frac{1}{\cancel{3}}\left[\frac{e^{-x/3}}{-1/\cancel{3}}\right]_1^{\infty}$$

$$= -[e^{-\infty} - e^{-1/3}]$$

$$\boxed{P(x>1) = 0.7165}$$

(ii) $P(x<3) = \frac{1}{3}\int_0^{3} e^{-x/3}\,dx$

$$= \frac{1}{\cancel{3}}\left[\frac{e^{-x/3}}{-1/\cancel{3}}\right]_0^{3}$$

$$= -[e^{-1} - e^{0}]$$

$$\boxed{P(x<3) = 0.6321}$$

*** MQP

* 18. If the milage (in thousands of miles) of a certain
MQP radial tyre in a random variable with exponential distribution having mean 40,000 miles determine probability that the tyre will last

(i) Atleast 20,000 miles

(ii) Atmost 30,000 miles

sol $f(x) = \begin{cases} \alpha e^{-\alpha x} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$

$$\text{Mean} = \frac{1}{\alpha}$$

$$40,000 = \frac{1}{\alpha}$$

$$\boxed{\alpha = \frac{1}{40,000}}$$

$$f(x) = \begin{cases} \frac{1}{40,000} e^{-x/40,000} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

(i) $P(x \geq 20,000)$

$$= \frac{1}{40,000} \int_{20,000}^{\infty} e^{-x/40,000}$$

$$= \frac{1}{40,000} \left. \frac{e^{-x/40,000}}{-1/40,000} \right]_{20,000}^{\infty}$$

$$= -[e^{-\infty} - e^{-1/2}]$$

$$\boxed{P(x \geq 20,000) = 0.6065}$$

(ii) $P(x \leq 30,000)$

$$= \frac{1}{40,000} \int_{0}^{30,000} e^{-x/40,000}$$

$$= \frac{1}{40,000} \left. \frac{e^{-x/40,000}}{-1/40,000} \right]_{0}^{30,000}$$

$$= -[e^{-3/4} - e^{0}]$$

$$\boxed{P(x \leq 30,000) = 0.5276}$$

* 19. The sales per day in a shop is exponentially distributed with the average sale amounting to rupees ₹100 and the net profit is 8%, find the probability that the net profit exceeds ₹30 on 2 consecutive days

sol

$$\text{Mean} = \frac{1}{\alpha}$$

$$100 = \frac{1}{\alpha}$$

$$\alpha = \frac{1}{100}$$

$$\boxed{\alpha = 0.01}$$

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$f(x) = \begin{cases} 0.01\, e^{-0.01x} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

Let 'A' be amount of sales for which net profit is 8%.

$$\frac{8}{100} \times A = 30$$

$$A = \frac{30 \times 100}{8}$$

$$\boxed{A = ₹375}$$

$$P(\text{Profit} > 30) = P(\text{Sales} > 375)$$

$$= 0.01 \int_{375}^{\infty} e^{-0.01x}\, dx$$

$$= 0.01 \left[\frac{e^{-0.01x}}{-0.01} \right]_{375}^{\infty}$$

$$= -\left[0 - e^{-3.75} \right]$$

$$P(\text{Profit} > 30) = e^{-3.75} \quad \text{, one day}$$

For 2 consecutive days

$P(\text{Profit} > 30) = e^{-3.75} \times e^{-3.75}$

$\boxed{P(\text{Profit} > 30) = 0.0006}$

MODULE 2 NOTES

MATHEMATICS FOR COMPUTER SCIENCE		SEMESTER	3
Course Code	BCS301	CIE Marks	50
Teaching Hours/Week(L:T:P:S)	3:2:0:0	SEE Marks	50
Total Hours of Pedagogy	40 Hours Theory + 20 Hours Tutorial	Total Marks	100
Credits	4	Exam Hours	3hrs
Examination type (SEE)	Theory		
Course Outcomes CO3: Apply the notion of a discrete-time Markov chain and n-step transition probabilities to solve the given problem.			

Module 2: Joint probability distribution and Markov chain	(12 hours)
Joint probability distribution: Joint Probability distribution for two discrete random variables, expectation, covariance and correlation. Markov Chain: Introduction to Stochastic Process, Probability Vectors, Stochastic matrices, Regular stochastic matrices, Markov chains, Higher transition probabilities, Stationary distribution of Regular Markov chains and absorbing states.	

Joint Probability distributions

The distributions associated with two random variables is referred to as joint distributions.

If X and Y are 2 discrete random variables, we define the joint probability function of X and Y by $P(X = x_i, Y = y_j) = f(x_i, y_j)$ or J_{ij} where $f(x, y)$ satisfy the conditions

a) $f(x, y) \geq 0$

b) $\sum_x \sum_y f(x, y) = 1$

Joint Probability Table

X \ Y	y_1	y_2	----	y_n	Sum
x_1	J_{11}	J_{12}	----	J_{1n}	$f(x_1)$
x_2	J_{21}	J_{22}	----	J_{2n}	$f(x_2)$
--	--	--	--	--	--
--	--	--	--	--	--
x_m	J_{m1}	J_{m2}	----	J_{mn}	$f(x_m)$
Sum	$g(y_1)$	$g(y_2)$	----	$g(y_n)$	1

Here $\{f(x_1), f(x_2) ---- f(x_m)\}$ & $\{g(y_1), g(y_2) --- g(y_n)\}$ are called marginal probability distributions of X & Y

<u>Note</u> :- i) $f(x_1) + f(x_2) + --- + f(x_m) = 1$

$g(y_1) + g(y_2) + -- + g(y_n) = 1$

ii) $\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i\, y_j) = \sum_{1}^{m} \sum_{1}^{n} J_{ij} = \sum_{1}^{m} \sum_{1}^{n} P(x_i, y_j) = 1$

Independent random variables

X & Y are ind random variables if $P(X=x_i, Y=y_j)$
$= P(X=x_i) \cdot P(Y=y_j)$

ie $f(x_i)\, g(y_j) = J_{ij}$

ie if each entry J_{ij} in the table is equal to the product of its marginal entries.

Expectation, Variance, Covariance and Correlation

Expectation of $X = E(X) = \mu_X = \sum_i x_i f(x_i)$

Expectation of $Y = E(Y) = \mu_Y = \sum_j y_j g(y_j)$

Expectation of $XY = E(XY) = \sum_{ij} x_i y_j J_{ij}$

Covariance $= \text{COV}(X,Y) = E(XY) - E(X) \cdot E(Y)$

Correlation $= \rho(X,Y) = \dfrac{\text{COV}(X,Y)}{\sigma_X \sigma_Y}$

Where $\sigma_X^2 = E(X^2) - \mu_X^2$ $\qquad$ $\sigma_Y^2 = E(Y^2) - \mu_Y^2$

and $E(X^2) = \sum_i x_i^2 f(x_i)$ $\qquad$ and $E(Y^2) = \sum_j y_j^2 g(y_j)$

This is called variance of X and Y.

Expectation of a random variable is the average of random variable.

Covariance is a term that quantifies the extent of dependence b/n 2 random variables.

MQP

1) The joint probability of 2 R.V's X & Y is as follows

X \ Y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

find marginal distribution of X & Y, Compute $E(X)$, $E(Y)$, $E(XY)$, σ_X, σ_Y, $COV(X,Y)$, $\rho(X,Y)$

Marginal distribution of X

x_i	1	5
$f(x_i)$	1/2	1/2

Marginal distribution of Y

y_j	-4	2	7
$g(y_j)$	3/8	3/8	1/4

a) $E(X) = \sum x_i f(x_i) = \mu_X$
$= 1(1/2) + 5(1/2)$
$= 3$

b) $E(Y) = \sum y_j g(y_j) = \mu_Y$
$= -4(3/8) + 2(3/8) + 7(1/4)$
$= 1$

c) $E(XY) = \sum x_i y_j J_{ij}$
$= 1(-4)(1/8) + 1(2)(1/4) + (1)(7)(1/8)$
$+ 5(-4)(1/4) + 5(2)(1/8) + 5(7)(1/8)$
$= 3/2$

$\sigma_x^2 = E(x^2) - \mu_x^2$

$E(x^2) = \sum x_i^2\, f(x_i)$

$= (1)^2(1/2) + (5)^2(1/2)$

$= 13$

$\sigma_x^2 = 13 - 9 = 4$

$\therefore \sigma_x = 2$

$\sigma_y^2 = E(y^2) - \mu_y^2$

$E(y^2) = \sum y_j^2\, g(y_j)$

$= (-4)^2(3/8) + (2)^2(3/8) + 7^2(1/4)$

$= 79/4$

$\sigma_y^2 = 79/4 - 1 = 75/4$

$\therefore \sigma_y = 4.33$

$\mathrm{cov}(x,y) = E(xy) - E(x)\cdot E(y) \;\big|\; E(xy) - \mu_x \cdot \mu_y$

$= 3/2 - 3(1)$

$= -3/20$

$\rho(x,y) = \dfrac{\mathrm{cov}(x,y)}{\sigma_x \sigma_y}$

$= \dfrac{-3/2}{2(4.33)} = \underline{\underline{-0.1732}}$

Jan 2024.

Q) The joint probability distribution table for 2 R.V's X & Y is as follows

X\Y	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

Determine marginal probability distributions of X & Y. Also find Expectation of X, Y & XY, S.D of X, Y, Covariance of X & Y, Correlation of X & Y. further V.T X & Y are dependent R.V's Also find $P(X+Y>0)$.

M.D of X

x_i	1	2
$f(x_i)$	0.6	0.4

M.D of Y

y_j	-2	-1	4	5
$g(y_j)$	0.3	0.3	0.1	0.3

a) $E(X) = \sum_i x_i f(x_i)$

$= 1(0.6) + 2(0.4) = 1.4$

b) $E(Y) = \sum_j y_j g(y_j)$

$= -2(0.3) + (-1)(0.3) + 4(0.1) + 5(0.3) = 1$

c) $E(XY) = \sum_{ij} x_i y_j J_{ij}$

$= 1(-2)(0.1) + 1(-1)(0.2) + 1(4)(0) + 1(5)(0.3)$

$+ 2(-2)(0.2) + 2(-1)(0.1) + 2(4)(0.1)$

$+ 2(5)(0) = 0.9$

$$\text{dy } \sigma_x^2 = E(x^2) - \mu_x^2 \qquad = 2.2 - (1.4)^2 = 0.24$$

$$E(x^2) = \sum_i x_i^2 f(x_i)$$

$$= 1(0.6) + 4(0.4) = 2.2$$

$$\text{ly } \sigma_y^2 = E(y^2) - \mu_y^2 \qquad = 10.6 - 1^2 = 9.6$$

$$E(y^2) = \sum_j y_j^2 g(y_j)$$

$$= 4(0.3) + 1(0.3) + 16(0.1) + 25(0.3)$$

$$= 10.6$$

$$\therefore \sigma_x = 0.49 \quad , \sigma_y = 3.1$$

$$\text{lly } \mathrm{cov}(x, y) = E(xy) - E(x)E(y)$$

$$= 0.9 - 1.4(1)$$

$$= -0.5$$

$$\text{ly } \rho(x, y) = \frac{\mathrm{cov}(x, y)}{\sigma_x \sigma_y} = \frac{-0.5}{0.49 \times 3.1} = -0.3$$

for x & y to be ind variables, $f(x_i)\, g(y_j) = J_{ij}$

But $f(x_1)\, g(y_1) = 0.6 \times 0.3 = 0.18$

$J_{11} = 0.1$

$\therefore f(x_1)\, g(y_1) \neq J_{11}$

$\Rightarrow$ x & y are dependent R.v's

$x = \{x_1, x_2\} = \{1, 2\}$

$y = \{y_1, y_2, y_3, y_4\} = \{-2, -1, 4, 5\}$

$J_{11} = 0.1$, $J_{12} = 0.2$, $J_{13} = 0$, $J_{14} = 0.3$, $J_{21} = 0.2$, $J_{22} = 0.1$, $J_{23} = 0.1$, $J_{24} = 0$

$X+Y>0$ is possible when (X, Y) takes the values

$(x_1\ y_3) = (1, 4)$; $(x_1\ y_4) = (1, 5)$; $(x_2\ y_2) = (2, -1)$;

$(x_2\ y_3) = (2, 4)$; $(x_2\ y_4) = (2, 5)$

$\therefore\ P(X+Y>0) = J_{13} + J_{14} + J_{22} + J_{23} + J_{24}$

$= 0.5$

Model paper

3) A fair coin is tossed thrice. The R.V's X & Y are defined as follows. X = 0 or 1 according as head or tail occurs on the first toss

Y = Number of heads.

a) Determine the distributions of X & Y

b) Determine the joint distribution of X & Y

c) Obtain the expectation of X, Y & XY

Also find S.D of X & Y

d) Compute Covariance & Correlation of X & Y

S	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
X	0	0	0	0	1	1	1	1
Y	3	2	2	1	2	1	1	0

$X = \{0, 1\}$ $\qquad$ $Y = \{0, 1, 2, 3\}$

$P(X=0) = 4/8 = 1/2$ $\qquad$ $P(Y=0) = 1/8$ $\qquad$ $P(Y=2) = 3/8$

$P(X=1) = 4/8 = 1/2$ $\qquad$ $P(Y=1) = 3/8$ $\qquad$ $P(Y=3) = 1/8$

a) M.D of X & Y

x_i	0	1
$f(x_i)$	1/2	1/2

y_j	0	1	2	3
$g(y_j)$	1/8	3/8	3/8	1/8

b) J.D of X & Y

$J_{ij} = P(X = x_i, Y = y_j)$

$J_{11} = P(X=0, Y=0) = 0$ [∵ X=0 means head on first toss, Y=0 means no head which is not possible]

$J_{12} = P(X=0, Y=1) = 1/8$ [HTT]

$J_{13} = P(X=0, Y=2) = 2/8 = 1/4$ [HHT & HTH]

$J_{14} = P(X=0, Y=3) = 1/8$ [HHH]

$J_{21} = P(X=1, Y=0) = 1/8$ [TTT]

$J_{22} = P(X=1, Y=1) = \frac{2}{8} = 1/4$ [THT, TTH]

$J_{23} = P(X=1, Y=2) = 1/8$ [THH]

$J_{24} = P(X=1, Y=3) = 0$ [Impossible]

X \ Y	0	1	2	3	Sum
0	0	1/8	1/4	1/8	1/2
1	1/8	1/4	1/8	0	1/2
Sum	1/8	3/8	3/8	1/8	1

$$\mu_x = E(X) = \sum_i x_i f(x_i)$$
$$= 0\times 1/2 + 1\times 1/2 = 1/2$$

$$\mu_y = E(Y) = \sum_j y_j g(y_j)$$
$$= 0\times 1/8 + 1\times 3/8 + 2\times \frac{3}{8} + 3\times \frac{1}{8} = \frac{3}{2}$$

$$E(XY) = \sum_{ij} x_i y_j J_{ij}$$
$$= 0 + 0 + 1/4 + 2/8 + 0 = 1/2$$

$$\sigma_X^2 = E(X^2) - \mu_X^2$$
$$E(X^2) = \sum_i x_i^2 f(x_i)$$
$$= 0\times 1/2 + 1\times 1/2 = 1/2$$
$$\sigma_X^2 = 1/2 - 1/4 = 1/4$$
$$\sigma_X = 1/2$$

$$\sigma_Y^2 = E(Y^2) - \mu_Y^2$$
$$E(Y^2) = \sum_j y_j^2 g(y_j)$$
$$= 0\times 1/8 + 1\times 3/8 + 4\times \frac{3}{8} + 9\times \frac{1}{8}$$
$$= 3$$
$$\sigma_Y^2 = 3 - 9/4 = 3/4$$
$$\sigma_Y = 0.8660$$

$$\text{cov}(X,Y) = E(XY) - E(X)E(Y)$$
$$= \frac{1}{2} - \frac{3}{4} = -1/4$$

$$\rho(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{-1/4}{\sqrt{3}/4} = -1/\sqrt{3} = -0.5774$$

4) Consider the joint distribution of X & Y given below

X \ Y	0	1	2	3	Sum
0	0	1/8	1/4	1/8	1/2
1	1/8	1/4	1/8	0	1/2
Sum	1/8	3/8	3/8	1/8	1

Compute

a) $P(X=1, Y=2)$

b) $P(X\geq 1, Y\leq 2)$

c) $P(X\leq 1, Y\leq 2)$

d) $P(X+Y \geq 2)$

$X=\{x_1\ x_2\}=\{0,1\}$

$Y=\{y_1\ y_2\ y_3\ y_4\}=\{0,1,2,3\}$

$J_{11}=0,\ J_{12}=1/8,\ J_{13}=1/4,\ J_{14}=1/8,\ J_{21}=1/8,\ J_{22}=1/4,\ J_{23}=1/8,$
$J_{24}=0$

a) $P(X=1, Y=2)=P(x_2, y_3)=J_{23}=1/8$

b) $P(X\geq 1, Y\leq 2)=P(x_2, y_1)+P(x_2\ y_2)+P(x_2\ y_3)$
$=J_{21}+J_{22}+J_{23}$
$=1/8+1/4+1/8=1/2$

c) $P(X\leq 1, Y\leq 2)=P(x_1\ y_1)+P(x_1\ y_2)+P(x_1\ y_3)+P(x_2\ y_1)$
$+P(x_2\ y_2)+P(x_2\ y_3)$
$=J_{11}+J_{12}+J_{13}+J_{21}+J_{22}+J_{23}$
$=0+1/8+1/4+1/8+1/4+1/8$
$=7/8$

d) $P(X+Y \geq 2)$

$= P(x_1 y_3) + P(x_1 y_4) + P(x_2, y_2) + P(x_2 y_3) + P(x_2 y_4)$

$= J_{13} + J_{14} + J_{22} + J_{23} + J_{24}$

$= 1/4 + 1/8 + 1/4 + 1/8 + 0$

$= 3/4$

Model paper

5) Let X be a random variable with the following distribution & Y is defined to be X^2

x_i	-2	-1	1	2
$f(x_i)$	$1/4$	$1/4$	$1/4$	$1/4$

Determine

a) the distribution of of Y

b) the joint distribution of X & Y

c) $E(X)$ $E(Y)$ $E(XY)$

d) $\text{cov}(X, Y)$ & $\rho(X, Y)$

Given $y = x^2$

When $x = -2$ $y = 4$ $x = 2$, $y = 4$

$x = -1$ $y = 1$ $x = 1$, $y = 1$

a)

y_j	1	4
$g(y_j)$	$1/2$	$1/2$

Distribution of Y

b)

$X \backslash Y$	1	4
-2	0	$1/4$
-1	$1/4$	0
1	$1/4$	0
2	0	$1/4$

joint distribution of X & Y

c) $E(X) = \sum_i x_i f(x_i)$

$$= -2\left(\tfrac{1}{4}\right) + -1\left(\tfrac{1}{4}\right) + 1\left(\tfrac{1}{4}\right) + 2\left(\tfrac{1}{4}\right) = 0$$

d) $E(Y) = \sum_j y_j g(y_j)$

$$= 1\left(\tfrac{1}{2}\right) + 4\left(\tfrac{1}{2}\right) = 5/2$$

e) $E(XY) = \sum_{ij} x_i y_j J_{ij}$

$$= (-2\times1\times0) + (-2\times4\times\tfrac{1}{4}) + (-\tfrac{1}{4}) + 0 + \tfrac{1}{4} + 0 + 0 + 2$$

$$= 0$$

f) $\text{cov}(X,Y) = E(XY) - E(X)E(Y)$

$$= 0 - 0 = 0$$

g) $\rho(X,Y) = \dfrac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} = \underline{\underline{0}}$

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6) The joint Probability distribution of two discrete random variables x & y is given by $f(x,y) = K(2x+y)$ where x & y are integers such that $0 \leq x \leq 2$ & $0 \leq y \leq 3$

a) find the value of 'K'

b) find the M.D of X & Y

c) S.T random variables X & Y are dependent

(a)

X \ Y	0	1	2	3	$f(x)$
0	0	K	2K	3K	6K
1	2K	3K	4K	5K	14K
2	4K	5K	6K	7K	22K
$g(y)$	6K	9K	12K	15K	42K

$42K \rightarrow \sum_i \sum_j P(x_i y_j)$

WKT $\sum_i \sum_j P(x_i y_j) = 1$

$42K = 1$

$K = 1/42$

$\therefore$

X \ Y	0	1	2	3
0	0	1/42	2/42	3/42
1	2/42	3/42	4/42	5/42
2	4/42	5/42	6/42	7/42

(b) M.D of X

x_i	0	1	2
$f(x_i)$	1/7	1/3	11/21

M.D of Y

y_j	0	1	2	3
$g(y_j)$	1/7	3/14	2/7	5/14

(c) for X & Y to be independent variables

$f(x_i) \cdot g(y_j) = J_{ij}$

But $f(x_1) \cdot g(y_1) = f(0) \cdot g(0) = 1/7 \times 1/7 = 1/49$

where as $J_{11} = 0$

$f(x_1) \cdot g(y_1) \neq J_{11}$

Hence X & Y are dependent variables

Model QP

7) If the joint probability distribution of X & Y is given by

$f(x, y) = \frac{x+y}{30}$ for $x = 0, 1, 2, 3$ & $y = 0, 1, 2$

find a) $P[X \leq 2, Y = 1]$

b) $P[X > 2, Y \leq 1]$

c) $P[X > Y]$

X \ Y	0	1	2	f(x)
0	0	1/30	1/15	1/10
1	1/30	1/15	1/10	1/5
2	1/15	1/10	2/15	3/10
3	1/10	2/15	1/6	2/5
g(y)	1/5	1/3	7/15	1

a) $P[X \leq 2, Y = 1]$ $\left[\begin{matrix} X = \{0,1,2\} \\ Y = \{1\} \end{matrix}\right]$

$= P(0,1) + P(1,1) + P(2,1)$

$= J_{12} + J_{22} + J_{32}$

$= 1/30 + 1/15 + 1/10$

$= 1/5$

b) $P[X > 2, Y \leq 1]$ $\left[\begin{matrix} X = \{3\} \\ Y = \{0,1\} \end{matrix}\right]$

$= P(3,0) + P(3,1)$

$= J_{41} + J_{42}$

$= 1/10 + 2/15 = 7/30$

c) $P[X > Y]$

$= P(1,0) + P(2,0) + P(2,1) + P(3,0) + P(3,1) + P(3,2)$

$= J_{21} + J_{31} + J_{32} + J_{41} + J_{42} + J_{43}$

$= 1/30 + 1/15 + 1/10 + 1/10 + 2/15 + 1/6$

$= 3/5$

MQP

8) The joint probability distribution of 2 random variables X & Y is

Y \ X	-3	2	4
1	0.1	0.2	0.2
3	0.3	0.1	0.1

a) Are X & Y independent?

b) Evaluate $P[Y \leq 2]$

c) Evaluate $P[X+Y \leq 2]$

d) find M.D of X & Y

e) find $E(X)$ & $E(Y)$

Y \ X	-3	2	4	g(y)
1	0.1	0.2	0.2	0.5
3	0.3	0.1	0.1	0.5
f(x)	0.4	0.3	0.3	1

a) $f(x_i) \cdot g(y_j) = J_{ij}$, for X & Y to be ind R.V's

$f(x_1) \cdot g(y_1) = J_{11}$

$0.4 \times 0.5 = 0.1$

$0.2 \neq 0.1$

$\therefore f(x_i) \cdot g(y_j) \neq J_{ij}$

$\therefore$ X & Y are dep random variables

b) $P[Y \leq 2] = P(y=1) = 0.1 + 0.2 + 0.2 = 0.5$

c) $P[X+Y \leq 2] = P(x=-3, y=1) + P(x=-3, y=3)$

$= 0.1 + 0.3$

$= 0.4$

d) M.D of X

x_i	-3	2	4
$f(x_i)$	0.4	0.3	0.3

M.D of Y

y_j	1	3
$g(y_j)$	0.5	0.5

e) $E(X) = \sum x_i f(x_i)$

$= -3(0.4) + 2(0.3) + 4(0.3)$

$= 0.6$

$E(Y) = \sum y_j g(y_j)$

$= 1(0.5) + 3(0.5)$

$= 2$

MQP

9) Determine the value of k so that the function $f(x,y) = K|x-y|$ for $x = -2, 0, 2$ & $y = -2, 3$ represents joint probability distribution of the random variables X & Y. Also find $\text{cov}(X,Y)$

X \ Y	-2	3	$f(x)$
-2	0	5K	5K
0	2K	3K	5K
2	4K	K	5K
$g(y)$	6K	9K	15K

$\to \sum_i \sum_j P(x_i, y_j) = 1$

$\Rightarrow 15K = 1 \Rightarrow \boxed{K = 1/15}$

$\therefore$

X \ Y	-2	3
-2	0	1/3
0	2/15	1/5
2	4/15	1/15

M.D of X

x_i	-2	0	2
$f(x_i)$	$1/3$	$1/3$	$1/3$

M.D of Y

y_j	-2	3
$g(y_j)$	$2/5$	$3/5$

$$E(X) = \sum_i x_i f(x_i) = -2(1/3) + 0 + 2(1/3) = 0$$

$$E(Y) = \sum_j y_j g(y_j) = -2(2/5) + 3(3/5) = 1$$

$$E(XY) = \sum x_i y_j J_{ij} = (-2\times -2\times 0) + (-2\times 3\times 1/3) + 0 + 0$$
$$+ 2\times -2\times \frac{4}{15} + 2\times 3\times \frac{1}{15}$$
$$= -8/3$$

$$COV(X,Y) = E(XY) - E(X)\,E(Y) = -8/3 - 0 = -8/3$$

QP

10) Random Variables X & Y have foll joint distribution

X \ Y	-3	2	4
1	0.1	0.2	0.2
2	0.3	0.1	0.1

find a) M.D of X & Y

b) COV (X, Y)

c) Correlation of X & Y

d) Evaluate $P[y \leq 2]$, $P[x + y \leq 2]$

e) Are the variables X, Y independent random variable?

a) M.D of X

x_i	1	2
$f(x_i)$	0.5	0.5

M.D of Y

y_j	-3	2	4
$g(y_j)$	0.4	0.3	0.3

b) $E(X) = \sum_i x_i f(x_i) = 1(0.5) + 2(0.5) = 1.5$

$E(Y) = \sum_j y_j g(y_j) = -3(0.4) + 2(0.3) + 4(0.3) = 0.6$

$$E(XY) = \sum x_i y_j J_{ij}$$

$$= (1 \times -3 \times 0.1) + (1 \times 2 \times 0.2) + (1 \times 4 \times 0.2) + (2 \times -3 \times 0.3) + (2 \times 2 \times 0.1) + (2 \times 4 \times 0.1)$$

$$= 0.3$$

$$Cov(XY) = E(XY) - E(X)E(Y)$$
$$= 0.3 - (1.5)(0.6)$$
$$= -0.6$$

c) $\sigma_X^2 = E(X^2) - [E(X)]^2$ $\qquad$ $\sigma_Y^2 = E(Y^2) - [E(Y)]^2$

$E(X^2) = \sum x_i^2 f(x_i)$
$= 1^2 \times 0.5 + 2^2 \times 0.5$
$= 2.5$

$E(Y^2) = \sum y_j^2 g(y_j)$
$= 3^2(0.4) + 2^2(0.3) + 4^2(0.3)$
$= 9.6$

$\sigma_X^2 = 2.5 - (1.5)^2$
$= 0.25$

$\sigma_X = 0.5$

$\sigma_Y^2 = 9.6 - (0.6)^2$
$= 9.24$

$\sigma_Y = 3.0397$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{-0.6}{0.5 \times 3.0397} = -0.3948$$

d) $P(y \leq 2) = P(y=-3) + P(y=2)$
$= 0.4 + 0.3 = 0.7$

$P(x+y \leq 2) = P(x=1, y=-3) + P(x=2, y=-3)$
$= 0.1 + 0.3 = 0.4$

e) $f(x_1) \cdot g(y_1) = 0.5 \times 0.4 = 0.2$

$J_{11} = 0.1$

$\therefore f(x_i) \cdot g(y_j) \neq J_{ij}$ $\qquad$ $\therefore$ X & Y are dependent R.V's

11) If X & Y are independent random variables X taking values 2, 5, 7 with the probability 1/2, 1/4, 1/4 respectively & Y taking values 3, 4, 5 with the probability 1/3, 1/3, 1/3

find a) Joint probability distribution

b) S.T $COV(X, Y) = 0$

a) M.D of X

x_i	2	5	7
$f(x_i)$	1/2	1/4	1/4

M.D of Y

y_j	3	4	5
$g(y_j)$	1/3	1/3	1/3

Given X & Y are ind R.V's

$\therefore J_{ij} = f(x_i)\, g(y_j)$

$J_{11} = f(x_1)\, g(y_1) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

$J_{12} = f(x_1)\, g(y_2) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

$J_{13} = f(x_1)\, g(y_3) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

$J_{21} = f(x_2)\, g(y_1) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

$J_{22} = f(x_2)\, g(y_2) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

$J_{23} = f(x_2)\, g(y_3) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

$J_{31} = f(x_3)\, g(y_1) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

$J_{32} = f(x_3)\, g(y_2) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

$J_{33} = f(x_3)\, g(y_3) = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

Joint Probability distribution

X \ Y	3	4	5	$f(x)$
2	1/6	1/6	1/6	3/6
5	1/12	1/12	1/12	3/12
7	1/12	1/12	1/12	3/12
$g(y)$	4/12	4/12	4/12	1

b) $E(X) = \sum_i x_i f(x_i) = 2(1/2) + 5(1/4) + 7(1/4) = 4$

$E(Y) = \sum_j y_j g(y_j) = 3(1/3) + 4(1/3) + 5(1/3) = 4$

$E(XY) = \sum x_i y_j J_{ij}$

$= (2\times3\times1/6) + (2\times4\times1/6) + (2\times5\times1/6) + (5\times3\times1/12) + (5\times4\times1/12) + (5\times5\times1/12) + (7\times3\times1/12) + (7\times4\times1/12) + (7\times5\times1/12)$

$= 16$

$\text{cov}(XY) = E(XY) - E(X)E(Y)$

$= 16 - 4(4)$

$= 0$

QP

12) Given the foll joint distribution of R.V's X & Y find the corresponding M.D of X & Y. Compute covariance & correlation of the random variables X & Y.

X \ Y	1	3	9
2	1/8	1/24	1/12
4	1/4	1/4	0
6	1/8	1/24	1/12

M.D of X

x_i	2	4	6
$f(x_i)$	1/4	1/2	1/4

M.D of Y

y_j	1	3	9
$g(y_j)$	1/2	1/3	1/6

$$E(X) = \sum_i x_i f(x_i)$$
$$= 2(1/4) + 4(1/2) + 6(1/4) = 4$$

$$E(Y) = \sum_j y_j g(y_j)$$
$$= 1(1/2) + 3(1/3) + 9(1/6) = 3$$

$$E(XY) = \sum x_i y_j J_{ij}$$
$$= (2\times1\times1/8) + (2\times3\times1/24) + (2\times9\times1/12) +$$
$$(4\times1\times1/4) + (4\times3\times1/4) + (4\times9\times0) +$$
$$(6\times1\times1/8) + (6\times3\times1/24) + (6\times9\times1/12)$$
$$= 12$$

$$\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y)$$
$$= 12 - 4\times3$$
$$= 0$$

$$\sigma_X^2 = E(X^2) - [E(X)]^2$$
$$E(X^2) = \sum x_i^2 f(x_i)$$
$$= 2^2(1/4) + 4^2(1/2) + 6^2(1/4)$$
$$= 18$$

$$\sigma_Y^2 = E(Y^2) - [E(\ )]^2$$
$$E(Y^2) = \sum y_j^2 g(y_j)$$
$$= 1^2(1/2) + 3^2(1/3) + 9^2(1/6)$$
$$= 17$$

$$\sigma_X^2 = 18 - 4^2 = 2$$
$$\sigma_X = \sqrt{2}$$

$$\sigma_Y^2 = 17 - 3^2 = 8$$
$$\sigma_Y = \sqrt{8}$$

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma_X\,\sigma_Y} = \frac{0}{\sqrt{2}\sqrt{8}} = 0$$

Stochastic Process

A real variable X associated with every outcome is called a random variable or stochastic variable.

The family of all such random variables namely the set $\{X(t), t \in T\}$ is called stochastic process defined on sample space S with 't' as parameter.

Here $X_0 = X(0)$ is called as the initial state of the system.

The values assumed by the random variable $X(t)$ are called states and the set of all possible values forms the state space of the process.

If the state space of a stochastic process is discrete | continuous then it is called a discrete | continuous state process or chain.

MQP

Probability Vector

A vector $v = (v_1, v_2 - - - v_n)$ is called a probability vector if each one of its components are non negative & their sum is equal to unity.

Eg:- $u = (1, 0)$ $\quad$ $v = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$

MQP

Stochastic matrix

A square matrix $P = p_{ij}$ having every row in the form of a probability vector is called a stochastic matrix $[\text{ie } p_{ij} \geq 0 \text{ \& } \sum p_{ij} = 1, \forall i]$

Eg:- $\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

MQP

Regular stochastic matrix

A stochastic matrix P is said to be a regular stochastic matrix if all the entries of some power P^n are positive. Eg:- $A = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$

$\therefore$ A is a regular stochastic matrix $[n = 2]$

1) which vectors are probability vectors? No neg entry sum = 1

a) $(1/4, 3/2, -1/4, 1/2)$
Not a probability vector because negative entry

b) $(5/2, 0, 8/3, 1/6, 1/6)$
Not a prob vector because sum is not equal to 1

c) $(1/12, 1/2, 1/6, 0, 1/4)$
is a prob vector because all entries are non-negative & sum = 1

d) $(3, 0, 2, 5, 3)$

$3+0+2+5+3 = 13$

$\therefore \left(\frac{3}{13}, 0, \frac{2}{13}, \frac{5}{13}, \frac{3}{13}\right)$

is a prob vector because all entries are non-negative and sum $= 1$.

2) Which matrices are stochastic?

[square matrix
No neg entry
sum of each row $= 1$]

a) $\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

Not stochastic because it is not a square matrix.

b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Stochastic because it is a square matrix, non-neg, sum of each row $= 1$.

c) $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}$

not stochastic because sum of each row $\neq 1$ (last row)

d) $\begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$

No stochastic as we have negative entry

3) Which of the stochastic matrices are regular?

[diagonal element should not be 1
No entries to be zero
No neg entries]

a) $A = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$

Not regular since 1 lie on the main diagonal

b) $B = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$

$B^2 = B \cdot B = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 3/8 & 3/8 & 1/4 \end{pmatrix}$

$B^3 = B^2 \cdot B = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 7/16 & 7/16 & 1/8 \end{pmatrix}$

since entries of b_{13} b_{23} are zero, $\therefore$ it is not regular

c) $C = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$

$C^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix}$

$C^3 = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{pmatrix}$

$C^4 = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}$

$C^5 = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 1/8 & 1/2 & 3/8 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$

$\therefore$ All entries are +ve and non-zero.

C is regular stochastic matrix.

MQP

4) Find the unique fixed probability vector for the regular stochastic matrix

$$A = \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Let $v = (x, y, z)$ be the unique fixed prob vector associated with P

$\Rightarrow x + y + z = 1$ ——— (1)

We have by property of regular stochastic matrix

$vP = v$

$$[x \ y \ z] \begin{pmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{pmatrix} = (x, y, z)$$

$$\left(y/2 ,\ \frac{3x}{4} + \frac{y}{2} + z ,\ x/4 \right) = (x, y, z)$$

$y = 2x$, $\quad \frac{3x}{4} + \frac{y}{2} + z = y$, $\quad x = 4z$

$3x + 2y + 4z - 4y = 0$

$3x - 2y + 4z = 0$

from (1)

$z = 1 - x - y$

$= 1 - x - 2x$

$= 1 - 3x$

$\therefore 3x - 2y + 4z = 0$

$3x - 2(2x) + 4(1 - 3x) = 0$

$3x - 4x + 4 - 12x = 0$

$\Rightarrow x = 4/13$

$y = 8/13$, $z = 1/13$

Hence the reqd unique fixed probability vector is $v = (4/13, 8/13, 1/13)$

Jan 24

5) S.T $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is a regular stochastic matrix. Also find the associated unique fixed probability vector.

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^3 = P \cdot P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$P^4 = P \cdot P^3 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/2 \end{bmatrix}$$

$$P^5 = P \cdot P^4 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$$

$\therefore$ P is a regular stochastic matrix

Let $v = (x, y, z)$ be the unique fixed prob vector associated with P

$\Rightarrow x + y + z = 1$ ———— (1)

We have $vP = v$

$$(x\ y\ z) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = (x\ y\ z)$$

$$\left(z/2 \ , \ x+\frac{z}{2} \ , \ y\right) = (x \ y \ z)$$

$$\frac{z}{2} = x \ , \quad x+\frac{z}{2} = y \ , \quad y = z$$

$$z = 2x \qquad y = z = 2x$$

from (1)

$$x + y + z = 1$$

$$x + 2x + 2x = 1$$

$$x = 1/5$$

$$y = z = 2/5$$

$\therefore v = (1/5 \ , \ 2/5 \ , \ 2/5)$ is the reqd u.f.p. vector

Jan 24

Q) find the unique fixed probability vector for the regular stochastic matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

Let $v = (x, y, z)$ be the unique fixed prob vector associated with P

$\Rightarrow x + y + z = 1$ ——— (1)

We have $vP = v$

$$(x \ y \ z) \begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix} = (x \ y \ z)$$

$$\left(\frac{y}{6} \ , \ x+\frac{y}{2}+\frac{2z}{3} \ , \ \frac{y}{3}+\frac{z}{3}\right) = (x, y, z)$$

$$\frac{y}{6} = x \ , \quad x+\frac{y}{2}+\frac{2z}{3} = y \ , \quad \frac{y}{3}+\frac{z}{3} = z$$

$y = 6x \quad , \quad 6x - 3y + 4z = 0 \quad , \quad y = 2z$

$6x = 2z$

$z = 3x$

from (1)

$x + 6x + 3x = 1$

$x = 1/10$

$y = 6/10 \quad , \quad z = 3/10$

$\therefore \; v = \left(1/10, 6/10, 3/10\right)$ is the reqd u. p. vector

MQP

7) S.T the matrix $\begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$ is a regular stochastic matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$P^3 = P \cdot P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^4 = P \cdot P^3 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$P^5 = P \cdot P^4 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/8 & 1/2 & 3/8 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$\therefore$ P is a regular stochastic matrix

Markov Chains

Let the outcomes $x_1, x_2, \ldots$ of a sequence of trials satisfy the foll properties

a) Each outcome belong to the finite set of the outcomes $\{a_1, a_2, \ldots a_m\}$.

b) The outcome of any trial depend atmost upon the outcome of the immediate preceding trial.

Probability p_{ij} is associated with every pair of states (a_i, a_j) that a_j occurs immediately after a_i occurs. Such a stochastic process is called a finite Markov chain. p_{ij} are called transition probabilities & they form a square matrix of order 'm' called the transition probability matrix (t.p.m) denoted by P.

$$P = [p_{ij}] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & & & \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

The transition matrix of a Markov chain is a stochastic matrix.

$\left[\because\ 0 \leq p_{ij} \leq 1 \ \&\ \sum p_{ij} = 1\right]$

Q.P

Higher transition probabilities

The probability that the system changes from the state a_i to the state a_j in exactly 'n' steps is denoted by $P_{ij}^{(n)}$

ie $a_i \to a_{r_1} \to a_{r_2} \to ----\to a_{r_{n-1}} \to a_j$

The matrix formed by $P_{ij}^{(n)}$ is called the n-step transition matrix

$$[P^{(n)}] = [P_{ij}^{(n)}]$$

Let $P^{(0)} = [P_1^{(0)}, P_2^{(0)} ---- P_n^{(0)}]$ denote the initial probability distribution at the start of the process & let $P^{(n)} = [P_1^{(n)}, P_2^{(n)} ---- P_m^{(n)}]$ denote the n^{th} step probability distribution at the end of n steps.

ie $P^{(1)} = P^{(0)} \cdot P$, $P^{(2)} = P^{(0)} \cdot P^2$ ------ $P^{(n)} = P^{(0)} P^n$.

MQP

Stationary distribution of regular Markov chains

A Markov chain is said to be regular if the associated transition probability matrix 'P' is regular.

If P is a regular stochastic matrix of the Markov chain, then the sequence of n step transition matrices $P^2, P^3 ---P^n$ approaches the matrix V whose rows are each the unique fixed probability vector V of P.

$p^{(n)} = p^{(0)} P^n \quad \left[p^{(n)} = \{ p_1^{(n)}, p_2^{(n)} \cdots p^{(n)} \} \right]$

further as $n \to \infty$, $p_i^{(n)} = v_i$ where $i = 1, 2 \cdots m$

This is called the stationary distribution of the markov chain & $v = (v_1, v_2 \cdots v_m)$ is called stationary (fixed) probability vector of the Markov chain.

QP

Irreducible

A Markov chain is said to be irreducible if every state can be reached from every other state in a finite no of steps.

MQP

Absorbing state of a Markov chain

In a Markov chain the process reaches to a certain state after which it continues to remain in the same state. Such a state is called an absorbing state of a Markov chain.

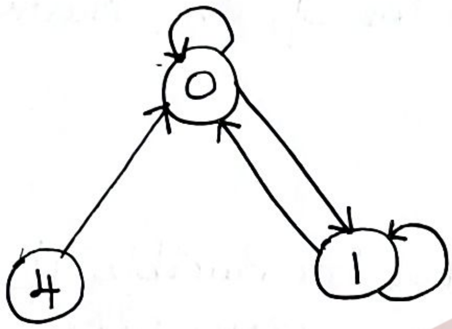
In this state, $p_{ij} = 1$ for $i = j$ & $p_{ij} = 0$ otherwise

Thus a state a_i of the Markov chain is absorbing if the i^{th} row of the t.p.m has 1 on the principal diagonal & zeroes elsewhere.

Eg:- $P = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix} \end{array}$

The states a_3 & a_4 are absorbing

1) Explain a) Transient state
b) Recurrent state
c) Periodic state
d) absorbing state of Markov chain



$$\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & \times & \times & 0 & 0 & 0 \\ 1 & \times & \times & 0 & 0 & 0 \\ 2 & 0 & 0 & \times & 0 & 0 \\ 3 & 0 & 0 & \times & \times & 0 \\ 4 & \times & 0 & 0 & 0 & 0 \end{array}$$

a) A state is transient if there's a nonzero probability that the chain will leave this state and never return to it. Here 3 & 4 are transient states.

Eg:- like a temporary city you might pass through once but not necessarily return

from 4 you can go to 0 but never back to 4
from 3 you can go to 2 & 3. once you go to 2 you're stuck (absorbing)

b) A state is recurrent if anytime that state is left it is possible to return back to that state in future with probability 1. Here 0 & 1 are recurrent states

Eg:- like your home city you'll always return eventually.

If you start at 0 you can bounce b/n 0 & 1.
Prob of returning to 0 starting at 0 is 1.

c) A state is periodic if the chain can only return to that state at multiples of some integer

starting from state 0 we only return to 0 at times $n = 2, 4 - - - -$ is a periodic state with period 2.

Eg:- you return home only at fixed times (every 2 days)

from 0 you go to 1 from 1 you go to 0
You can return to 0 only in 2 steps ($0 \rightarrow 1 \rightarrow 0$) $\therefore$ period = 2

Def: A state is absorbing if once entered it cannot be left.

Here state 2 is absorbing state. (from 2 you stay in 2 with prob. 1. Once you reach 2 you never leave so 2 is absorbing)

Eg:- like a dead end once you're there you're stuck forever

Q. The t.p.m of a Markov chain is given by

$$P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

and the initial probability distribution is $p^{(0)} = (1/2, 1/2, 0)$. find $P_{13}^{(2)}$, $P_{23}^{(2)}$, $p^{(2)}$ & $p_1^{(2)}$

$$P^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} = \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix}$$

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

$P_{13}^{(2)} = 3/8$ [It is the probability of moving from state a_1 to a_3 in 2 steps]

$P_{23}^{(2)} = 1/2$ [a_2 to a_3 in 2 steps]

$$p^{(2)} = p^{(0)} P^2 = [1/2, 1/2, 0] \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix}$$

$$= [\underset{P_1}{7/16}, \underset{P_2}{1/8}, \underset{P_3}{7/16}]$$

$\therefore p_1^{(2)} = 7/16$

Model QP

3) P.T the Markov chain whose t.p.m is

$$P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$ is irreducible. find the corresponding stationary probability vector | fixed probability vector.

$$P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$P^2 = P \cdot P = \frac{1}{6}\begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix} \cdot \frac{1}{6}\begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}$$

$$= \frac{1}{36}\begin{bmatrix} 18 & 6 & 12 \\ 9 & 21 & 6 \\ 9 & 12 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/4 & 7/12 & 1/6 \\ 1/4 & 1/3 & 5/12 \end{bmatrix}$$

Since all the entries in P^2 are +ve, t.p.m 'p' is regular.

∴ The Markov chain is irreducible.

Let $v = (x, y, z)$ be the unique fixed prob vector associated with P.

$$x + y + z = 1 \quad \text{——— (1)}$$

We also have $vP = v$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}$$

$$\left[y/2 + z/2 ,\ 2x/3 + z/2 ,\ x/3 + y/2 \right] = [x, y, z]$$

$$\frac{y}{2} + \frac{z}{2} = x \quad , \quad \frac{2x}{3} + \frac{z}{2} = y \quad , \quad \frac{x}{3} + \frac{y}{2} = z$$

$$y + z = 2x \qquad 4x + 3z = 6y \qquad 2x + 3y = 6z$$

from ①

$$x + 2x = 1$$

$$3x = 1$$

$$x = 1/3$$

Subs $x = 1/3$ in the other 2 eq^n

$$4(1/3) + 3z = 6y \quad , \quad 2(1/3) + 3y = 6z$$

$$4/3 + 3z = 6y \qquad 2/3 + 3y = 6z$$

$$4 + 9z = 18y \qquad 2 + 9y = 18z$$

$$\left.\begin{array}{l} 18y - 9z = 4 \\ -9y + 18z = 2 \end{array}\right\} \text{On solving} \quad z = 8/27$$

$$y = 10/27$$

$$\therefore \ v = (x, y, z) = (1/3, 10/27, 8/27)$$

Model QP

4) Every year a man trades for his car for a new car. If he has Maruti, he trade it for a ford. If he has a ford, he trade it for a hyundai. However if he has a hyundai, he is just as likely to trade it for a new hyundai as to trade it for a Maruthi or a Ford. In 2014, he bought his first car which was a hyundai. find the probability that he has

a) 2016 ford

b) 2016 hyundai

c) 2016 Maruthi

The state space of the system is {Maruthi (M), Ford (F), Hyundai (H)}

The associated transition matrix is

$$P = \begin{array}{c} \\ M \\ F \\ H \end{array} \begin{array}{c} \begin{array}{ccc} M & F & H \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array}$$

$$p^{(0)} = \begin{array}{c} \begin{array}{ccc} M & F & H \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{array}$$ {since in 2014 his first car was hyundai}

With 2014 as the first year 2016 is 2 years after. $\therefore P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix}$ — $\begin{bmatrix} P_{MM}^{(2)} & P_{MF}^{(2)} & P_{MH}^{(2)} \\ P_{FM}^{(2)} & P_{FF}^{(2)} & P_{FH}^{(2)} \\ P_{HM}^{(2)} & P_{HF}^{(2)} & P_{HH}^{(2)} \end{bmatrix}$

$$p^{(2)} = p^{(0)} \cdot P^2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix} = \begin{bmatrix} 1/9, & 4/9, & 4/9 \end{bmatrix}$$

a) 2016 ford = Hyundai to ford = $a_{32}^{(2)} = \frac{4}{9}$

b) 2016 hyundai = " to hyundai = $a_{33}^{(2)} = \frac{4}{9}$

c) 2016 Maruthi = " to maruthi = $a_{31}^{(2)} = \frac{1}{9}$

imp

5) A software engineer goes to his workplace everyday by motorbike or by car. He never goes by bike on 2 consecutive days but if he goes by car on a day he is equally like to go by car or by bike on the next day. find the transition matrix for the chain of the mode of transport he uses. If car is used on the first day of the week find the probability that

a) bike is used

b) car is used after 4 days (or on fifth day)

Here state space of the system is

$S = \{\text{motor bike } (B), \text{ Car } (C)\}$

The associated transition matrix is

$$P = \begin{array}{c c} & \begin{array}{cc} B & C \end{array} \\ \begin{array}{c} B \\ C \end{array} & \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

since the car is used on the first day, the prob that bike is used is zero and that car is used is 1. $\therefore p^{(0)} = (0, 1)$

The probability distribution of the mode of transport after 4 days is given by

$$p^{(4)} = p^{(0)}\, p^{4}$$

$$p^{4} = \begin{pmatrix} 3/8 & 5/8 \\ 5/16 & 11/16 \end{pmatrix}$$

$$\therefore p^{(4)} = (0, 1)\begin{pmatrix} 3/8 & 5/8 \\ 5/16 & 11/16 \end{pmatrix}$$

$$= \Big(\underset{p^{(4)}_{Bike}}{5/16},\ \underset{p^{(4)}_{Car}}{11/16}\Big)$$

After 4 days (or on fifth day) prob of using a bike $= 5/16$ & prob of using a car $= 11/16$

Jan 24

Q) Three boys A, B, C are throwing ball to each other. A always throws the ball to B & B always throws the ball to C. C is just as likely to throw the ball to B as to A. If C was the first person to throw the ball find the probabilities that after three throws

a) A has the ball

b) B has the ball

c) C has the ball

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & 1/2 & 1/2 & 0 \end{array}$$

Initially if C has the ball, the associated initial probability vector is $p^{(0)} = (0, 0, 1)$

$$p^{(3)} = p^{(0)} P^3 = (0,0,1) \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$= \left(\underset{p_A^{(3)}}{1/4}, \underset{p_B^{(3)}}{1/4}, \underset{p_C^{(3)}}{1/2}\right)$$

a) A has the ball $= p_A^{(3)} = 1/4$

b) B has the ball $= p_B^{(3)} = 1/4$

c) C has the ball $= p_C^{(3)} = 1/2$

7) A gamblers luck follows a pattern. If he wins a game, the probability of winning the next game is 0.6. However if he loses a game, the probability of losing the next game is 0.7. There is an even chance of gambler winning the first game. If so

a) what is the prob of he winning the second game?

b) What is the prob of he winning the third game?

c) In the long run, how often he will win?

$$P = \begin{array}{c c} & \begin{array}{cc} W & L \end{array} \\ \begin{array}{c} W \\ L \end{array} & \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} \end{array}$$

Probability of winning first game is $1/2$

Initial prob vector $p^{(0)} = (1/2, 1/2)$

a) $p^{(1)} = p^{(0)} \cdot P = (1/2, 1/2) \begin{pmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{pmatrix} = (\underset{P_W^{(1)}}{0.45}, \underset{P_L^{(1)}}{0.55})$

$\therefore$ prob of winning second game $= 0.45$

b) $p^{(2)} = p^{(0)} p^2 = (\underset{P_W^{(2)}}{0.435}, \underset{P_L^{(2)}}{0.565})$

$\therefore$ prob of winning third game $= 0.435$

c) $v = (x, y)$, $x + y = 1$ —— (1)

We have $vP = v$

$$[x \; y] \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} = [x, y]$$

$0.6x + 0.3y = x$ $\qquad$ $0.4x + 0.7y = y$

$0.3y = 0.4x$

$y = \frac{0.4}{0.3} x$

from eq (1)

$x + y = 1$

$x + \frac{0.4}{0.3} x = 1$

$0.7x = 0.3$

$x = 3/7$

$y = 1 - x$
$= 1 - 3/7$
$= 4/7$

$v = [3/7, 4/7] = [p^{(W)}, p^{(L)}]$

$\therefore$ In the long run he wins $3/7$ of the time imp Jan 24

8) Each year, a man trades his car for a new car in 3 brands of the popular company. If he has a swift he trades it for Dzire. If he has a dzire he trades it for a Wagon R. If he has a Wagon R he is just as likely to trade it for a new Wagon R or for a Dzire or a swift one. In 2020 he bought his first car which was Wagon R. Find the probability that he has

a) 2022 wagon R

b) 2022 swift

c) 2023 dzire

d) 2023 Wagon R

The state space of the system is $\{S, D, W\}$ where S is swift, D is Dzire, W is wagon R.

The associated transition matrix is

$$P = \begin{array}{c} \\ S \\ D \\ W \end{array} \begin{array}{c} \begin{array}{ccc} S & D & W \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array}$$

$p^{(0)} = [0, 0, 1]$ as his first car was Wagon R in 2020.

With 2020 as the first year 2022 is 2 years later.

$$p^{(2)} = p^{(0)} P^2 = [0, 0, 1] \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{ccc} p_S^{(2)} & p_D^{(2)} & p_W^{(2)} \end{array} \\ [1/9, 4/9, 4/9] \end{array}$$

a) 2022 wagon R $= 4/9$

b) 2022 swift $= 1/9$

With 2020 as first year, 2023 is after 3 years

$$p^{(3)} = p^{(0)} P^3 = [0, 0, 1] \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \\ 4/27 & 7/27 & 16/27 \end{bmatrix}$$

$$= \left[\overset{p_S^{(3)}}{4/27}, \overset{p_D^{(3)}}{7/27}, \overset{p_W^{(3)}}{\frac{16}{27}} \right]$$

c) 2023 dezire $= 7/27$

d) 2023 Wagon R $= 16/27$

MQP

Q) Three boys A, B, C are throwing ball to each other. A is just as likely to throw the ball to B as to C. B always throws the ball to A and C is just as likely to throw the ball to A as to B. Find the probability that C has the ball after three throws if now A has the ball.

$$P = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

Initially A has the ball. The associated initial probability vector is $p^{(0)} = [1, 0, 0]$

Since C has the ball after 3 throws,

$$p^{(3)} = p^{(0)} P^3 = [1 \quad 0 \quad 0] \begin{bmatrix} 1/4 & 3/8 & 3/8 \\ 3/4 & 1/4 & 0 \\ 3/8 & 3/8 & 1/4 \end{bmatrix}$$

$$= [\underset{p_A^{(3)}}{1/4}, \underset{p_B^{(3)}}{3/8}, \underset{p_C^{(3)}}{3/8}]$$

$\therefore$ Probability that C has the ball after 3 throws is $p_C^{(3)} = 3/8$

10) The students study habits are as follows. If he studies on one night he is 60% sure not to study on next night. On the other hand, if he does not study on to night he is 80% sure to study next night. Write the transition probability matrix for his chain of study. In the long run how often he studies on monday night. What is the probability that he does not study on Friday night?

The state space of the system is {S = student studies, T = student does not study}

$$P = \begin{array}{c} \\ S \\ T \end{array} \begin{array}{c} \begin{array}{cc} S & T \end{array} \\ \begin{bmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{bmatrix} \end{array}$$

Since he studies on monday night is taken

$$p^{(0)} = [1 \quad 0]$$

Friday is four days after Monday

$$p^{(4)} = p^{(0)} P^4$$

$$= (1, 0)\begin{pmatrix} 0.5824 & 0.4176 \\ 0.5568 & 0.4432 \end{pmatrix}$$

$$= (\overset{p_S^{(4)}}{0.5824}, \overset{p_T^{(4)}}{0.4176})$$

Probability that the student does not study on Friday night is 0.4176.

Let $v = (x, y)$, $x + y = 1$ — (1)

We have $vP = v$

$$(x, y)\begin{pmatrix} 0.4 & 0.6 \\ 0.8 & 0.2 \end{pmatrix} = (x, y)$$

In long run,

$$0.4x + 0.8y = x$$
$$0.6x + 0.2y = y$$

from (1) $y = 1 - x$

$$\therefore 0.4x + 0.8(1 - x) = x$$
$$0.4x + 0.8 - 0.8x = x$$
$$0.8 = 1.4x$$
$$x = 0.5714, \quad y = 1 - 0.5714 = 0.4286$$

$\therefore$ In the long run, student studies 57.14% of the time.

MQP

11) A salesman territory consists of 3 cities A, B & C. He never sells in the same city on successive days. If he sells in city A then the next day he sells in city B. However if he sells in either B or C then the next day he is twice as likely to sell in city A as in other city. In long run, how often does he sell in each of the cities.

Let A, B, C be the 3 cities as given

$$P = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 2/3 & 0 & 1/3 \\ C & 2/3 & 1/3 & 0 \end{array}$$

Let $v = (x, y, z)$, $x+y+z=1$ —— (1)

We have $vP = v$

$$(x, y, z)\begin{pmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{pmatrix} = (x, y, z)$$

$$\left[\frac{2}{3}y + \frac{2}{3}z, \quad x + \frac{1}{3}z, \quad y/3\right] = [x, y, z]$$

$$\frac{2}{3}y + \frac{2}{3}z = x, \quad x + \frac{1}{3}z = y, \quad y/3 = z$$

$$y = 3z$$

$$\frac{2}{3}(3z) + \frac{2}{3}(z) = x$$

$$2z + \frac{2}{3}z = x$$

$$8z = 3x \quad \therefore x = \frac{8}{3}z$$

We have $x + y + z = 1$

$$\frac{8}{3}z + 3z + z = 1$$

$$20z = 3$$

$$z = 3/20$$

$$x = \frac{8}{3}z = \frac{8}{3}\left(\frac{3}{20}\right) = 8/20 = 2/5$$

$$y = 3z = 3\left(3/20\right) = 9/20$$

In long run in city

$A \rightarrow x \rightarrow \frac{2}{5} \times 100 = 40\%$

$B \rightarrow y \rightarrow 9/20 \times 100 = 45\%$

$C \rightarrow z \rightarrow \frac{3}{20} \times 100 = 15\%$

Hence in long run, salesman sells 40%, 45% & 15% in cities A, B & C.

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12) Three boys X, Y, Z are throwing a ball to each other. X always throws the ball to Y & Y always throws the ball to Z. But Z is just as likely to throw the ball to Y or as to X. Write t.p.m. If Z is the first person to throw the ball find the prob that X has the ball after fourth

throw ?

Given x, y, z be three boys.

$$P = \begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

Initially, z throws the ball, the associated initial probability vector is $p^{(0)} = (0, 0, 1)$

Since it is to be taken after 4 throws,

$$p^{(4)} = p^{(0)} P^4 = (0, 0, 1) \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}$$

$$= \left(\overset{p_x^{(4)}}{1/4}, \overset{p_y^{(4)}}{1/2}, \overset{p_z^{(4)}}{1/4} \right)$$

Probability that x has the ball after fourth throw $= 1/4$

MQP

3) Every year a man trades his car for a new car. If he has a Maruthi, he trades it for an ambassador. If he has an ambassador, he trades it for a Santro. However, if he has a Santro, he is just as likely to trade it for a new Santro as to trade it for a Maruthi or an Ambassador. In 2000, he bought his first car which was a santro. Find the probability ~~that~~ he has

a) 2002 Santro b) 2002 Maruthi

c) 2003 Ambassador d) 2003 Santro

The state space of the system is $\{M, A, S\}$ where M - maruti, A - ambassador, S - Santro

The associated transition matrix is

$$P = \begin{array}{c} \\ M \\ A \\ S \end{array} \begin{array}{c} \begin{array}{ccc} M & A & S \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array}$$

Since his first car was Santro the initial probability is $p^{(0)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

With 2000 as the first year, 2002 is 2 years later

$$p^{(2)} = p^{(0)} \cdot P^2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix}$$

$$= \begin{bmatrix} \overset{p_M^{(2)}}{1/9}, & \overset{p_A^{(2)}}{4/9}, & \overset{p_S^{(2)}}{4/9} \end{bmatrix}$$

a) Prob that he has 2002 Santro $= 4/9$

b) Prob that he has 2002 Maruthi $= 1/9$

With 2000 as the first year, 2003 is 3 years later

$$p^{(3)} = p^{(0)} \cdot P^3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \\ 4/27 & 7/27 & 16/27 \end{bmatrix}$$

$$= \begin{bmatrix} \overset{p_M^{(3)}}{4/27}, & \overset{p_A^{(3)}}{7/27}, & \overset{p_S^{(3)}}{16/27} \end{bmatrix}$$

$\therefore$ Prob that he has 2003 ambassador $= 7/27$

$\therefore$ Prob that he has 2003 santro $= 16/27$