(Module I) PROBABILITY Distributions

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Random variables:

In a random experiment, if a real variable is associated with every outcome then it is called a random variable or stochastic variable. It is denoted by X, Y, Z ... . The set of all real numbers of a random variable X is called the range of X.

$X: S \rightarrow R$, $S \rightarrow$ Sample Space

Ex:- $X \rightarrow$ no of heads

$Y \rightarrow$ no of tails

Outcome :	HH	HT	TH	TT
(R.V) X :	2	1	1	0
Y :	0	1	1	2

Discrete & Continuous Random variables:-

If a R.V takes finite or countably infinite number of values then it is called discrete random variable.

Ex:- Tossing a coin & observing outcome

If a R.V takes non countable infinite number of values then it is called a non discrete or continuous random variable. It can assume any value in the interval of real numbers.

Ex:- Weight of articles

Discrete Probability distribution:-

If for each value x_i of a discrete random variable X, a real no $p(x_i)$ is assigned such that

(i) $p(x_i) \geq 0$ (ii) $\sum_i p(x_i) = 1$,

then the function $p(x)$ is called a probability function.

If the probability that X take the values x_i is p_i, then

$P(X = x_i) = p_i$ or $p(x_i)$.

The set of values $[x_i, p(x_i)]$ is called a discrete (finite) probability distribution of D.R.V 'X'. The function $p(x)$ is called probability Mass function (p.m.f)

The distribution function $f(x)$ defined by $f(x) = P(X \leq x) = \sum_{i=1}^{x} p(x_i)$, x being an integer is called cumulative distribution function (c.d.f). For discrete probability distribution

*** Mean $(\mu) = \sum_i x_i \, p(x_i)$

Variance $(V) = \sum_i (x_i - \mu)^2 . p(x_i)$.

Standard deviation $(\sigma) = \sqrt{V}$

[OR Variance $= \sum x_i^2 p(x_i) - \left[\sum_i x_i \, p(x_i)\right]^2$]

(1) Find mean & variance for the following probability distribution choosing K suitably.

x:	0	1	2	3	4	5
$p(x)$:	K	5K	10K	10K	5K	K

Soln:- (i) $p(x) \geq 0$ for all x

(ii) $\Sigma p(x) = 1$

$$K + 5K + 10K + 10K + 5K + K = 1$$

$$32K = 1 \Longrightarrow \boxed{K = \frac{1}{32}}$$

$$\text{Mean} = \mu = \Sigma x p(x)$$

$$= 0 \times \frac{1}{32} + 1 \times \frac{5}{32} + 2 \times \frac{10}{32} + 3 \times \frac{10}{32} + 4 \times \frac{5}{32} + 5 \times \frac{1}{32}$$

$$\boxed{\text{Mean} = 2.5}$$

$$\text{variance} = \Sigma (x_i - \mu)^2 p(x_i)$$

$$= \Sigma (x_i - 2.5)^2 p(x_i)$$

$$= (0-2.5)^2 \times \frac{1}{32} + (1-2.5)^2 \times \frac{5}{32} + (2-2.5)^2 \times \frac{10}{32} + (3-2.5)^2 \times \frac{10}{32} + (4-2.5)^2 \times \frac{5}{32} + (5-2.5)^2 \times \frac{1}{32}.$$

$$\therefore \boxed{\text{variance} = 1.25}$$

(2) *** 6M

The probability distribution of a finite random variable X is given by following table : (1)

x_i :	-2	-1	0	1	2	3
$p(x_i)$:	0.1	k	0.2	$2k$	0.3	k

Determine value of k & find the mean, variance & s.d.

soln:-

$$\Sigma p(x_i) = 1$$

$$0.1 + k + 0.2 + 2k + 0.3 + k = 1$$

$$4k = 1 - 0.6$$

$$\boxed{k = 0.1}$$

$$\text{Mean} = \mu = \Sigma x_i p(x_i)$$

$$= -0.2 - 0.1 + 0 + 0.2 + 0.6 + 0.3$$

$$\boxed{\text{Mean} = 0.8}$$

$$\text{Variance} = \sigma^2 = \Sigma (x_i - \mu)^2 p(x_i)$$

$$= \Sigma (x_i - 0.8)^2 p(x_i)$$

$$= (-2-0.8)^2 \times 0.1 + (-1-0.8)^2 \times 0.1 + (0-0.8)^2 \times 0.2 + (1-0.8)^2 \times 0.2 + (2-0.8)^2 \times 0.3 + (3-0.8)^2 \times 0.1$$

$$\boxed{\text{Variance} = 2.16}$$

$$S.D = \sqrt{V} = \sqrt{2.16}$$

$$\boxed{S.D = 1.4697}$$

*** MQP/Dec 2023

3) The probability distribution of finite R.V 'X' is given by

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Find K, $P(x<6)$, $P(x\geq 6)$, $P(3<x\leq 6)$, $P(0<x<5)$

Soln:- W.K.T $\sum p(x_i) = 1$

$$9K + 10K^2 = 1$$

$$10K^2 + 9K - 1 = 0$$

$$K = \frac{1}{10} \quad \text{or} \quad K = -1$$

But $K \neq -1$ $[\because p(x_i) \geq 0]$

$$\therefore \boxed{K = \frac{1}{10}}$$

Probability distribution:

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

$$P(x<6) = 1 - P(x\geq 6)$$

$$= 1 - [P(x=6) + P(x=7)]$$

$$= 1 - \left[\frac{2}{100} + \frac{17}{100}\right]$$

$$\boxed{P(x<6) = 0.81}$$

$$P(x \geq 6) = P(x=6) + P(x=7)$$

$$= \frac{2}{100} + \frac{17}{100} \Rightarrow \boxed{P(x \geq 6) = 0.19}$$

$$P(3 < x \leq 6) = P(x=4) + P(x=5) + P(x=6)$$

$$= \frac{3}{10} + \frac{1}{100} + \frac{2}{100} \Rightarrow \boxed{P(3 < x \leq 6) = 0.33}$$

$$P(0 < x < 5) = P(x=1) + P(x=2) + P(x=3) + P(x=4)$$

$$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} \Rightarrow \boxed{P(0 < x < 5) = 0.8}$$

*** MQP

4) The p.d.f of x is given by

x :	0	1	2	3	4	5	6
$p(x)$:	K	3K	5K	7K	9K	11K	13K

Find K. Also find $P(x \geq 5)$, $P(3 < x \leq 6)$, $P(x < 4)$, $P(3 < x < 6)$, $P(x > 1)$ & Mean of x or $E(x)$

Soln: W.K.T $\sum p(x_i) = 1$

$$49K = 1 \Rightarrow \boxed{K = \frac{1}{49}}$$

Probability distribution,

x :	0	1	2	3	4	5	6
$p(x)$:	$\frac{1}{49}$	$\frac{3}{49}$	$\frac{5}{49}$	$\frac{7}{49}$	$\frac{9}{49}$	$\frac{11}{49}$	$\frac{13}{49}$
$x.p(x)$:	0	$\frac{3}{49}$	$\frac{10}{49}$	$\frac{21}{49}$	$\frac{36}{49}$	$\frac{55}{49}$	$\frac{78}{49}$

$$P(x \geq 5) = P(x=5) + P(x=6)$$

$$= \frac{11}{49} + \frac{13}{49} \Rightarrow \boxed{P(x \geq 5) = \frac{24}{49} = 0.4898}$$

$$P(3 < x \leq 6) = P(x=4) + P(x=5) + P(x=6)$$

$$= \frac{9}{49} + \frac{11}{49} + \frac{13}{49}$$

$$\boxed{P(3 < x \leq 6) = \frac{33}{49} = 0.6735}$$

$$P(x < 4) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$$

$$= \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49}$$

$$\boxed{P(x<4) = \frac{16}{49} = 0.3265}$$

$$P(3 < x < 6) = P(x=4) + P(x=5)$$

$$= \frac{9}{49} + \frac{11}{49}$$

$$\boxed{P(3 < x < 6) = \frac{20}{49} = 0.4082}$$

$$P(x > 1) = 1 - P(x \leq 1)$$

$$= 1 - [P(x=0) + P(x=1)]$$

$$= 1 - \left[\frac{1}{49} + \frac{3}{49}\right]$$

$$\boxed{P(x > 1) = \frac{45}{49} = 0.9184}$$

$$\text{Mean } (\mu) = \sum x\, p(x)$$

$$= \frac{1}{49}\left[0 + 3 + 10 + 21 + 36 + 55 + 78\right]$$

$$= \frac{203}{49} \Rightarrow \boxed{\text{Mean} = \frac{29}{7} = 4.1428}$$

(5) If the R.V 'x' take the values 1, 2, 3, 4 such that,

$2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$,

find p.d.f & c.d.f of X

Soln:

X:	1	2	3	4
$P(x)$:	p_1	p_2	p_3	p_4

Given $2p_1 = 3p_2 = p_3 = 5p_4$

W.K.T $\sum p(x_i) = 1$

$p_1 + p_2 + p_3 + p_4 = 1$

$p_1 + \frac{2}{3}p_1 + 2p_1 + \frac{2}{5}p_1 = 1$

$$\boxed{p_1 = \frac{15}{61}}$$

$\boxed{p_2 = \frac{10}{61}}$ $\boxed{p_3 = \frac{30}{61}}$, $\boxed{p_4 = \frac{6}{61}}$

$X = x_i$:	1	2	3	4
$p(x)$:	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$
$f(x) = \Sigma p(x_i)$:	$\frac{15}{61}$	$\frac{25}{61}$	$\frac{55}{61}$	$\frac{61}{61} = 1$

⑥ Find value of K such that $p(x) = \frac{K}{2^x}$; $x = 1, 2, 3 \ldots$ represents probability distribution.

Soln:

$$\Sigma p(x) = 1$$

$$\sum_{x=1}^{\infty} \frac{K}{2^x} = 1$$

$$\frac{K}{2^1} + \frac{K}{2^2} + \ldots = 1$$

$$K\left[\frac{1}{2} + \frac{1}{2^2} + \ldots\right] = 1$$

$\left[\frac{1}{2} + \frac{1}{2^2} + \ldots \text{ is a G.P. } a = \frac{1}{2} \quad \boxed{r = \frac{1}{2}} \quad S_\infty = \frac{a}{1-r}\right]$

$$K\left[\frac{\frac{1}{2}}{1 - \frac{1}{2}}\right] = 1 \Longrightarrow \boxed{K = 1}$$

7) A random variable X take the values $-3, -2, -1, 0, 1, 2, 3$ such that,

$P(x=0) = P(x<0)$ and

$P(x=-3) = P(x=-2) = P(x=-1) = P(x=1) = P(x=2) = P(x=3)$.

Find the probability distribution.

Soln: Let the probability distribution $(x, p(x))$ be

x	-3	-2	-1	0	1	2	3
$p(x)$	P_1	P_2	P_3	P_4	P_5	P_6	P_7

Given, $P(x=0) = P(x<0)$

$P(x=0) = P(x=-1) + P(x=-2) + P(x=-3)$

$P_4 = P_3 + P_2 + P_1 \rightarrow (1)$

Given,

$P(x=-3) = P(x=-2) = P(x=-1) = P(x=1) = P(x=2) = P(x=3)$

$P_1 = P_2 = P_3 = P_5 = P_6 = P_7 \rightarrow (2)$

W.K.T $\sum p(x_i) = 1$.

$P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_7 = 1 \rightarrow (3)$

(2) in (1) $\Rightarrow$ $P_4 = 3P_1 \rightarrow (4)$

(2) in (3) $\Rightarrow$ $6P_1 + P_4 = 1$

$6P_1 + 3P_1 = 1$ [Using (4)]

$$9P_1 = 1 \Rightarrow \boxed{P_1 = \frac{1}{9}}$$

$$P_4 = 3P_1 = 3 \times \frac{1}{9} \Rightarrow \boxed{P_4 = \frac{1}{3}}$$

Probability distribution is given by

x	-3	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

8) The pdf of a random variable x is given by the following table:

x	-3	-2	-1	0	1	2	3
$P(x)$	K	$2K$	$3K$	$4K$	$3K$	$2K$	K

Find (i) The value of K

(ii) $P(x>1)$

(iii) $P(-1<x<2)$

(iv) mean of x

(v) S.D of x

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Soln: W.K.T $\sum P(x_i) = 1$

$$K + 2K + 3K + 4K + 3K + 2K + K = 1$$

$$16K = 1$$

$$\Rightarrow \boxed{K = \frac{1}{16}}$$

x	-3	-2	-1	0	1	2	3
$P(x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$
$(x-0)^2$	9	4	1	0	1	4	9

(i) $P(x>1) = P(x=2) + P(x=3)$

$$= \frac{2}{16} + \frac{1}{16} = \frac{3}{16}.$$

(ii) $P(-1 < x < 2) = P(x=0) + P(x=1) + P(x=2)$

$$= \frac{4}{16} + \frac{3}{16} + \frac{2}{16} = \frac{9}{16}$$

(iii) Mean $= \mu = \Sigma x\, P(x)$

$$= \frac{1}{16}\left[-3-4-3+0+3+4+3\right] = 0.$$

(iv) Variance $= V = \Sigma (x-\mu)^2 . P(x)$

$$= \frac{1}{16}\left[9+8+3+0+3+8+9\right] = \frac{40}{16} = \frac{5}{2}$$

$\Longrightarrow \boxed{V = 2.5}$

$$S.D = \sqrt{V} = \sqrt{\frac{5}{2}} = 1.5811$$

9) The Pdf of a random variable x is given by the following table:

x	-2	-1	0	1	2	3	4
$P(x)$	0.1	0.1	K	0.1	2K	K	K

(3)

Find (i) Value of K (ii) mean or $E(x)$ (iii) S.D.

Soln (i) W.K.T $\sum P(x_i) = 1$

$0.1 + 0.1 + K + 0.1 + 2K + K + K = 1$

$0.3 + 5K = 1 \Rightarrow 5K = 0.7$

$\boxed{K = 0.14}$

x	-2	-1	0	1	2	3	4
$p(x)$	0.1	0.1	0.14	0.1	0.28	0.14	0.14
$(x-1.34)^2 \cdot p(x)$	1.1156	0.5476	0.2514	0.0116	0.1220	0.3858	0.9906

(ii) Mean $= \mu = \sum x\, p(x)$

$= -0.2 - 0.1 + 0 + 0.1 + 0.56 + 0.42 + 0.56$

$\boxed{\text{Mean} = 1.34}$

(iii) Variance $= V = \sum (x-\mu)^2 p(x)$

$= 1.1156 + 0.5476 + 0.2514 + 0.0116 + 0.1220 + 0.3858 + 0.9906$

$\boxed{V = 3.4246}$

$S.D = \sqrt{V}$

$S.D = \sqrt{3.4246}$

$\boxed{S.D = 1.8506}$

*** MQP

10) Determine the value of K, so that the function $f(x) = K(x^2+4)$ for $x = 0, 1, 2, 3$ can serve as a probability distribution of the discrete random variable X. Also find

(i) $P(0 < x \le 2)$

(ii) $P(x \ge 1)$

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Soln:-

x :	0	1	2	3
$f(x) = P(x)$:	4K	5K	8K	13K

W.K.T $\sum P(x_i) = 1$

$30K = 1 \Rightarrow \boxed{K = \frac{1}{30}}$

x :	0	1	2	3
$P(x)$:	$\frac{4}{30}$	$\frac{5}{30}$	$\frac{8}{30}$	$\frac{13}{30}$

(i) $P(0 < x \le 2) = P(1) + P(2)$

$= \frac{5}{30} + \frac{8}{30} = \frac{13}{30} = 0.4333$

(ii) $P(x \ge 1) = 1 - P(x < 1) = 1 - P(0)$

$= 1 - \frac{4}{30} = \frac{26}{30} = 0.8667$

*** MQP

11) A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers find the probability distribution for the number of defectives. Find the mean & variance of this distribution. (10)

Soln: Defective → 3, Working → 5

No. of Defective computers 'x' :	0	1	2
P(x) :	$\frac{5C_2}{8C_2}$	$\frac{3C_1 \times 5C_1}{8C_2}$	$\frac{3C_2}{8C_2}$

Probability distribn:

x :	0	1	2
P(x) :	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$
$(x-0.75)^2 : p(x)$	$\frac{45}{224}$	$\frac{15}{448}$	$\frac{75}{448}$

$$\text{Mean} = \mu = \Sigma x\, P(x) = 0 + \frac{15}{28} + \frac{6}{28}$$

$$\boxed{\mu = 0.75}$$

$$\text{Variance} = v = \Sigma (x-\mu)^2 P(x)$$

$$= \left[\frac{45}{224} + \frac{15}{448} + \frac{75}{448}\right] = \frac{45}{112}$$

$$\boxed{v = 0.4018}$$

Bernoulli trial

A random experiment with only 2 possible outcomes categorized as success & failure is called a Bernoulli trial where the probability of success p is same for each trial and probability of failure denoted by q is $(1-p)$.

i.e $q = 1-p$ or $p+q=1$

Binomial distribution :

If 'p' is the probability of success & 'q' is the probability of failure, the probability of 'x' successes out of 'n' trials is given by $p(x) = {}^nC_x p^x q^{n-x}$.

Probability distribution of $(x, p(x))$ where $x = 0, 1, 2 \ldots n$ is given by

x :	0	1	2 ----	n
$p(x)$:	q^n	${}^nC_1 q^{n-1} p$	${}^nC_2 q^{n-2} p^2$ ----	p^n

$$\sum p(x) = q^n + {}^nC_1 q^{n-1} p + {}^nC_2 q^{n-2} p^2 + \ldots p^n$$

$$= (q+p)^n \quad [\text{Using binomial expansion}]$$

$$= 1^n$$

$$\sum p(x) = 1$$

$\therefore$ $p(x)$ is a probability function

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Mean & S.D of Binomial distribution :-

$$\text{Mean } (\mu) = \sum_{x=0}^{n} x\, p(x)$$

(5) (4)

$$\mu = \sum_{x=0}^{n} x \cdot {}^nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \left[\frac{n!}{(n-x)!\, x!}\right] p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{\cancel{x} \times n \times (n-1)!}{(n-x)! \times \cancel{x} \times (x-1)!}\, p \cdot p^{x-1} \cdot q^{n-x}$$

$$\mu = np \sum_{x=1}^{n} \frac{(n-1)!}{(n-x+1-1)!\,(x-1)!} p^{x-1} q^{n-x+1-1}$$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{[(n-1)-(x-1)]!\,(x-1)!} p^{x-1} q^{[(n-1)-(x-1)]}$$

$$= np \sum_{x=1}^{n} {}^{n-1}C_{x-1}\, p^{x-1} q^{[(n-1)-(x-1)]}$$

$$= np\,(p+q)^{n-1} \quad \left[\because \sum p(x) = \sum {}^{n}C_x p^x q^{n-x} = (p+q)^n\right]$$

$$= np\,(1)^{n-1}$$

$$\therefore \boxed{\mu = np} \rightarrow E(X) \rightarrow \text{Expectation}$$

$$\text{Variance } (v) = \sum x^2 p(x) - \left[\sum x\, p(x)\right]^2$$

$$= \sum x^2 p(x) - \mu^2$$

$$v = \sum x^2 p(x) - n^2p^2 \longrightarrow (1)$$

$$\sum_{x=0}^{n} x^2 p(x) = \sum_{x=0}^{n} (x^2 - x + x)\, p(x)$$

$$= \sum_{x=0}^{n} x(x-1)\, p(x) + \sum_{x=0}^{n} x\, p(x)$$

$$= \sum_{x=0}^{n} x(x-1)\, {}^{n}C_x p^x q^{n-x} + \mu$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{(n-x)!\, x!} p^x q^{n-x} + np$$

$$= \sum_{x=0}^{n} \frac{x(x-1)\, n(n-1)(n-2)!}{(n-x)!\, x(x-1)(x-2)!} p^x q^{n-x} + np$$

$$= \sum_{x=2}^{n} \frac{n(n-1)(n-2)!}{(n-x+2-2)!\,(x-2)!} p^2 \cdot p^{x-2} \cdot q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)! \cdot p^{x-2} q^{n-x+2-2}}{[(n-2)-(x-2)]!\,(x-2)!} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} q^{[(n-2)-(x-2)]} + np$$

$$= n(n-1)p^2 (q+p)^{n-2} + np$$

$$\left[\because \sum p(x) = \sum {}^{n}C_x p^x q^{n-x} = (p+q)^n \right]$$

$$= n(n-1)p^2 (1)^{n-2} + np$$

$$\therefore \sum x^2 p(x) = n(n-1)p^2 + np \longrightarrow (2)$$

(2) in (1)

$$V = n(n-1)p^2 + np - n^2p^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$\boxed{V = npq}$$

$$\boxed{S.D = \sigma = \sqrt{V} = \sqrt{npq}}$$

Poisson Distribution :-

This is regarded as the limiting form of the binomial distribution when 'n' is very large ($n \to \infty$) & p, the probability of success is very small ($p \to 0$) so that np tends to a fixed finite constant say m.

$$\boxed{p(x) = \frac{m^x e^{-m}}{x!}}$$ is known as poisson distribution of the R.V 'x' & 'x' is called poisson variate

x :	0	1	2	3	----
$p(x)$:	e^{-m}	$\frac{me^{-m}}{1!}$	$\frac{m^2e^{-m}}{2!}$	$\frac{m^3e^{-m}}{3!}$	----

Here $p(x) \geq 0$

$$\sum_{x=0}^{\infty} p(x) = e^{-m}\left\{1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots\right\}$$

$$= e^{-m} \cdot e^{m} = \underline{\underline{1}} \Rightarrow p(x) \text{ is probability function}$$

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Mean & S.D of poisson distribution:-

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$$\text{Mean}(\mu) = \sum_{x=0}^{\infty} x\, p(x)$$

$$= \sum_{x=0}^{\infty} \frac{x \cdot m^x \cdot e^{-m}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{x\, m^x e^{-m}}{x(x-1)!} = \sum_{x=1}^{\infty} \frac{m^{x-1+1} \cdot e^{-m}}{(x-1)!}$$

$$= m e^{-m} \sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}$$

$$= m e^{-m} \left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \cdots \right]$$

$$= m e^{-m} e^{m}$$

$$\therefore \boxed{\mu = m}$$

$$\text{Variance}(V) = \sum x^2 p(x) - \left[\sum x\, p(x) \right]^2 \rightarrow (1)$$

Consider,

$$\sum x^2 p(x) = \sum (x^2 - x + x)\, p(x)$$

$$= \sum x(x-1)\, p(x) + \sum x\, p(x)$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{m^x e^{-m}}{x!} + m$$

$$\sum x^2 p(x) = \sum_{x=2}^{\infty} \frac{\cancel{x(x-1)}\, m^x e^{-m}}{\cancel{x(x-1)}(x-2)!} + m$$

$$\sum x^2 p(x) = m^2 e^{-m} \sum_{x=2}^{\infty} \frac{m^{x-2}}{(x-2)!} + m$$

$$= m^2 e^{-m} \left[1 + m + \frac{m^2}{2!} + \cdots\right] + m$$

$$= m^2 e^{-m} \cdot e^{m} + m$$

$$\therefore \sum x^2 p(x) = m^2 + m \longrightarrow (2)$$

(2) in (1),

$$v = \cancel{m^2} + m - \cancel{m^2}$$

$$\boxed{v = m}$$

$$\boxed{S.D = \sigma = \sqrt{v} = \sqrt{m}}$$

(1) The probability of germination of a seed in a packet of seeds is found to be 0.7. If 10 seeds are taken for experimenting on germination in a lab, find probability that

(i) 8 seeds germinate

(ii) atleast 8 seeds germinate

Soln:- $p = 0.7$, $n = 10$, $q = 1 - 0.7 = 0.3$

(i) $P(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

$$P(x=8) = {}^{10}C_{8} (0.7)^{8} (0.3)^{2} = 0.2335$$

(ii) $P(x \geq 8) = P(x=8) + P(x=9) + P(x=10)$

$= 0.3828$

---

(2) X is a binomially distributed R.V, if mean & variance of X are 2 & $\frac{3}{2}$ respectively, find the distribution function & binomial distribution.

Soln:- Mean $= np$ & variance $= npq$

$np = 2$ & $npq = \frac{3}{2}$

$2q = \frac{3}{2}$

$$\boxed{q = \frac{3}{4}}$$

$p = 1 - q = 1 - \frac{3}{4}$

$$\boxed{p = \frac{1}{4}}$$

$np = 2 \Rightarrow \frac{n}{4} = 2$

$\Rightarrow \boxed{n = 8}$

Probability function, $p(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

$p(x) = {}^{8}C_{x} \left(\frac{1}{4}\right)^{x} \left(\frac{3}{4}\right)^{8-x}$

x :	0	1	2	3	4	5	6	7	8
$p(x)$:	0.1001	0.2670	0.3114	0.2076	0.0865	0.0231	0.0038	0.0004	0

*** 7M

(3) A die is thrown 8 times. Find the probability that 3 falls

(i) Exactly 2 times

(ii) Atleast once

(iii) At the most 7 times

Soln: $p = P(3) = \frac{1}{6}$, $q = 1 - \frac{1}{6} = \frac{5}{6}$, $n = 8$

$$p(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$$

$$p(x) = {}^{8}C_{x} \left(\frac{1}{6}\right)^{x} \left(\frac{5}{6}\right)^{8-x}$$

(i) $P(x=2) = {}^{8}C_{2} \left(\frac{1}{6}\right)^{2} \left(\frac{5}{6}\right)^{8-2}$

$= 0.2605$

(ii) $P(x \geq 1) = 1 - P(x < 1)$

$= 1 - P(x=0)$

$= 1 - {}^{8}C_{0} \left(\frac{1}{6}\right)^{0} \left(\frac{5}{6}\right)^{8-0}$

$= 0.7674$

(iii) $P(x \leq 7) = 1 - P(x > 7)$

$= 1 - P(x=8)$

$= 1 - {}^{8}C_{8} \left(\frac{1}{6}\right)^{8} \left(\frac{5}{6}\right)^{0}$

$= 1$

4) In a quiz contest of answering 'Yes' or 'No' what is the probability of guessing atleast 6 answers correctly out of 10 questions asked? Also find the probability of the same if there are 4 options for a correct answer. (7)

Soln: $p = \frac{1}{2}$, $q = 1-p = \frac{1}{2}$, $n = 10$

$$p(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= {}^{10}C_{x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{10-x} = {}^{10}C_{x} \left(\frac{1}{2}\right)^{10}$$

$$P(x \geq 6) = P(6) + P(7) + P(8) + P(9) + P(10)$$

$$= 0.2051 + 0.1172 + 0.0439 + 0.0098 + 0.0010$$

$$= 0.3770$$

If there are 4 options, $p = \frac{1}{4}$

$q = 1-p = \frac{3}{4}$, $n = 10$

$$p(x) = {}^{10}C_{x} \left(\frac{1}{4}\right)^{x} \left(\frac{3}{4}\right)^{10-x}$$

$$P(x \geq 6) = P(6) + P(7) + P(8) + P(9) + P(10)$$

$$= 0.0162 + 0.0031 + 0.0004 + 0 + 0$$

$$= 0.0197$$

5) The number of telephone lines busy at any instant of time is a binomial variate with probability that a line is busy is 0.1. If 10 lines are chosen at random, what is the probability that

(i) no line is busy.

(ii) atleast one line is busy

(iii) atmost two lines are busy

10

Soln:- $p = 0.1, \quad q = 1 - p = 0.9 \quad n = 10$

$$p(x) = n_{C_x} p^x q^{n-x} = 10_{C_x} (0.1)^x (0.9)^{10-x}$$

(i) $P(x=0) = 0.3487$

(ii) $P(x \geq 1) = 1 - P(x < 1)$

$= 1 - P(x=0) = 1 - 0.3487$

$= 0.6513$

(iii) $P(x \leq 2) = P(0) + P(1) + P(2)$

$= 0.3487 + 0.3874 + 0.1937$

$= 0.9298$

***(MQP)

(4) The probability that a pen manufactured by a factory is defective is 0.1. If 12 such pens are manufactured what is the probability that

(6)

(i) Exactly 2 are defective.

(ii) Atleast 2 are defective.

(iii) None of them are defective

(iv) Atleast 1 will be defective.

Soln: $p = 0.1, \quad q = 1 - p = 0.9, \quad n = 12$

$$p(x) = {}^{12}C_x (0.1)^x (0.9)^{12-x}$$

(i) $P(x=2) = {}^{12}C_2 (0.1)^2 (0.9)^{10} = \underline{\underline{0.2301}}$

(ii) $P(x \geq 2) = 1 - P(x < 2)$

$= 1 - [P(x=0) + P(x=1)]$

$= 1 - [0.2824 + 0.3766]$

$= \underline{\underline{0.341}}$

(iii) $P(x=0) = {}^{12}C_0 (0.1)^0 (0.9)^{12}$

$= \underline{\underline{0.2824}}$

(iv) $P(x \geq 1) = 1 - P(x < 1)$

$= 1 - [P(x=0)]$

$= 1 - 0.2824 = \underline{\underline{0.7176}}$

7) An airline knows that 5% of the people making reservations on a certain flight will not turn up. Consequently their policy is to sell 52 tickets for a flight that can only hold 50 people. What is the probability that there will be a seat for every passenger who turns up?

Soln: Let 'p' denote probability that a passenger will not turn up is

$p = 0.05 \quad q = 0.95$

Here x denotes number of passengers who will not turn up & $n = 52$

$P(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$

$P(x) = {}^{52}C_{x}\, (0.05)^{x}\, (0.95)^{52-x}$

For every passenger to have a seat, atleast 2 should not turn up.

$\therefore P(x \geq 2) = 1 - P(x < 2)$

$= 1 - [P(x=0) + P(x=1)]$

$= 0.7405$

(8) In 800 families with 5 children each, how many families would be expected to have,

(i) 3 boys

(ii) 5 girls

(iii) either 2 or 3 boys

(iv) atmost 2 girls by assuming probabilities for boys & girls to be equal.

(v) atleast 1 boy (vi) Atmost 2 boys.

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Soln: Let x denote no of boys

$n = 5, \quad p = \frac{1}{2}, \quad q = \frac{1}{2}$

$$P(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$$

$$= {}^{5}C_{x} \left(\frac{1}{2}\right)^{x} \left(\frac{1}{2}\right)^{5-x}$$

$$P(x) = \frac{{}^{5}C_{x}}{32}$$

Expected no in respect of 800 families

$$= 800\, P(x) = \frac{800 \times {}^{5}C_{x}}{32} = 25\left[{}^{5}C_{x}\right] = f(x)$$

(i) $f(3) = 250$

(ii) 5 girls mean 0 boy $\Rightarrow f(0) = 25$

(iii) $f(2) + f(3) = 500$

(iv) atmost 2 girls mean 5B & 0G or 4B & 1G or 3B & 2G $= f(5) + f(4) + f(3)$
$= 400$

(v) $f(1) + f(2) + f(3) + f(4) + f(5) = 125 + 250 + 250 + 125 + 25$
$\underline{\underline{= 775}}$

vi) $f(0) + f(1) + f(2) = 25 + 125 + 2$

(9) 4 coins are tossed 100 times & the following results are obtained. Fit a probability distribution for the data & calculate theoretical frequencies.

No of heads :	0	1	2	3	4
f :	5	29	36	25	5

Soln:- Let x denote no of heads

$$\text{Mean } (\mu) = \frac{\Sigma f x}{\Sigma f} = np$$

$$1.96 = 4p$$

$$\boxed{p = 0.49}$$

$$q = 1 - p = 0.51$$

$$p(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$p(x) = {}^{4}C_{x}\, (0.49)^{x} (0.51)^{4-x}$$

$$F(x) = 100\, p(x) = 100 \cdot {}^{4}C_{x} (0.49)^{x} (0.51)^{4-x}$$

$$F(0) = 7$$

$$F(1) = 26$$

$$F(2) = 37$$

$$F(3) = 24$$

$$F(4) = 6$$

(10) The probability of a shooter hitting a target is 1/3. How many times he should shoot so that the probability of hitting the target at least once is more than 3/4?

Soln:- $p = \frac{1}{3}$, $q = 1 - \frac{1}{3} = \frac{2}{3}$, $n = ?$

$$p(x) = {}^{n}C_{x} \left(\frac{1}{3}\right)^{x} \left(\frac{2}{3}\right)^{n-x}$$

$$P(x \geq 1) > \frac{3}{4}$$

$$1 - P(x < 1) > \frac{3}{4}$$

$$1 - P(x = 0) > \frac{3}{4}$$

$$1 - \left(\frac{2}{3}\right)^{n} > \frac{3}{4}$$

$$1 - \frac{3}{4} > \left(\frac{2}{3}\right)^{n}$$

$$\left(\frac{2}{3}\right)^{n} < \frac{1}{4}$$

$\therefore$ By inspection,

$\left(\frac{2}{3}\right)^{1} = 0.67$, $\left(\frac{2}{3}\right)^{2} = 0.44$

$\left(\frac{2}{3}\right)^{3} = 0.3$ $\left(\frac{2}{3}\right)^{4} = 0.2 < \frac{1}{4}$

$\therefore$ $\boxed{n = 4}$

(11) The no of accidents in a year to taxi drivers in a city follows a poisson distribution with mean 3. Out of 1000 taxi drivers, find approximately the no. of drivers with

(i) no accident in a year.

(ii) more than 3 accidents in a year

Soln:- $\mu = 3$

$\Rightarrow m = 3$ [$\because \mu = m$ for poisson's distribution]

$$P(x) = \frac{m^x e^{-m}}{x!} = \frac{3^x e^{-3}}{x!}$$

$$f(x) = 1000\,P(x) = \frac{1000 \times 3^x e^{-3}}{x!}$$

$$f(x) = \frac{50 \cdot 3^x}{x!}$$

(i) $f(0) = 50$

(ii) $P(x > 3) = 1 - P(x \leq 3) = 1 - [P(0) + P(1) + P(2) + P(3)]$

$= 1 - [0.0498 + 0.1494 + 0.2240 + 0.2240] = 0.3528$

$\therefore$ no of drivers $= 0.3528 \times 1000 \simeq 353$ drivers

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*** (MQP)

(12) If the probability of a bad reaction from a certain injection is 0.001, determine chance that out of 2000 individuals more than two will get a bad reaction. Also find chance that exactly 3 will suffer a bad reaction

Soln:- As $p = 0.001$ is very small. By poisson's distribution,

$$P(x) = \frac{m^x e^{-m}}{x!}$$

Mean $= m = np = 2000 \times 0.001 = 2$

$P(x > 2) = 1 - P(x \leq 2) = 1 - [P(x=0) + P(x=1) + P(x=2)]$

$= 1 - [0.1353 + 0.2707 + 0.2707]$

$\simeq 0.3233$

$\boxed{P(x=3) = 0.1804}$

*** MQP/Dec 2023 E

13)A

13) In certain factory turning out razor blades there is a small probability of $\frac{1}{500}$ for any blade to be defective. The blades are supplied in packets of 10, Find no of packets containing (i) one defective blade

(ii) two defective blades in a consignment of 10,000 packets.

(iii) No defective

9

Soln: $p = \frac{1}{500} = 0.002$, $n = 10$

$m = np = 10 \times 0.002 = 0.02$

$$P(x) = \frac{m^x e^{-m}}{x!} = \frac{(0.02)^x e^{-0.02}}{x!}$$

$$f(x) = 10,000 \times P(x)$$

$$= \frac{10,000 \times (0.02)^x e^{-0.02}}{x!}$$

(i) $f(1) = 196.04 \simeq 196$ packets

(ii) $f(2) = 1.96 \simeq 2$ packets.

(iii) $f(0) = 9802$ packets

14) Fit a Poisson distribution to the following set of observation

x	0	1	2	3	4
$f(x)$	122	60	15	2	1

Sol'n: Mean $= m = \dfrac{\sum x f(x)}{\sum f}$

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$$= \frac{0+60+30+6+4}{200}$$

$$\boxed{m = 0.5}$$

$$P(x) = \frac{m^x e^{-m}}{x!} = \frac{(0.5)^x e^{-0.5}}{x!}$$

15) Alpha Particles are emitted by a radioactive source at an average rate of 5 in 20 mins interval. Using Poisson distrib'n find probability that there will be (i) exactly 2 emissions (ii) atleast 2 emissions in randomly chosen 20 mins

Sol'n: $m = 5$, $p(x) = (5^x e^{-5})/x!$

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(i) $p(x=2) = 0.0842$

(ii) $p(x \geq 2) = 1 - p(x<2)$

$= 1 - [p(x=0) + p(x=1)]$

$= 1 - [0.0067 + 0.0337]$

$= 0.9596$

(15) If x is a poisson variate such that $P(x=2) = 9P(x=4) + 90P(x=6)$. Compute mean & variance of Poisson's distribution.

Soln:
$$P(x) = \frac{m^x e^{-m}}{x!}$$

Given, $P(x=2) = 9P(x=4) + 90P(x=6)$

$$\frac{m^2 e^{-m}}{2!} = \frac{9\,m^4 e^{-m}}{4!} + 90\,\frac{m^6 e^{-m}}{6!}$$

$$\frac{m^2}{2} = \frac{3}{8}m^4 + \frac{1}{8}m^6$$

$$m^4 + 3m^2 - 4 = 0 \Rightarrow m^4 + 4m^2 - m^2 - 4 = 0$$

$$m^2(m^2+4) - 1(m^2+4) = 0$$

$$(m^2-1)(m^2+4) = 0$$

$$m = \pm 1, \quad m = \pm 2i$$

Neglecting imaginary & negative values,

$$\boxed{m = 1} \qquad \therefore \boxed{\begin{array}{l} \text{Mean} = m = 1 \\ \text{Variance} = m = 1 \end{array}}$$

Continuous Probability distribution:-

p.d.f & c.d.f :-

$f(x)$ is said to be a p.d.f if

(i) $f(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

If X is a continuous R.V with p.d.f $f(x)$ then c.d.f of X is given by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

Mean & Variance:-

$$\text{Mean } (\mu) = \int_{-\infty}^{\infty} x\, f(x)\,dx = E(x)$$

$$\text{Variance } (\sigma^2) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

or $$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2 = E(x^2) - [E(x)]^2$$

(1) Exponential distribution:-

The continuous probability distrib'n having p.d.f $f(x)$ given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha > 0$ is known as exponential distribution.

Here $f(x) \geq 0$ &

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} \alpha e^{-\alpha x}\,dx$$

$$= \left[\frac{\cancel{\alpha} e^{-\alpha x}}{-\cancel{\alpha}}\right]_{0}^{\infty} = -[0-1] = \underline{\underline{1}}$$

$\therefore$ $f(x)$ is p.d.f

Note :-

for exponential distribution

$$\mu = \frac{1}{\alpha}, \quad V = \frac{1}{\alpha^2} \quad \& \quad \sigma = \frac{1}{\alpha}$$

Normal distribution:-

The continuous probability distribution having p.d.f, $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$,

where $-\infty < x < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$

is known as normal distribution.

Note :- 1) The graph of probability function $f(x)$ is a bell shaped curve symmetrical about the line $x = \mu$ & is called normal probability curve.

2) For normal distribution, Mean $= \mu$

Variance $= \sigma^2$, S.D $= \sigma$

(5) Find which of the following is a p.d.f

(i) $f_1(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

(ii) $f_2(x) = \begin{cases} 2x, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

Soln:- (i) $f_1(x) \geq 0$

$$\int_{-\infty}^{\infty} f_1(x) = \int_0^1 2x\,dx = \frac{2x^2}{2}\Big]_0^1 = 1$$

$\therefore$ $f_1(x)$ is a p.d.f

(ii) $f_2(x) = \begin{cases} 2x, & -1 < x < 0 \\ 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

In $-1 < x < 0$, $f_2(x) = 2x < 0$.

$\therefore$ $f_2(x) \not\geq 0$

$\therefore$ $f_2(x)$ is not a p.d.f

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(6) Find the value of 'K' such that the function, $f(x) = \begin{cases} Kx^2, & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$ is a p.d.f of a continuous R.V. Also find $P(1.5 < x < 2.5)$

Sol'n:-

$$f(x) \geq 0$$

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_{1}^{3} Kx^2\, dx = 1$$

$$\left.\frac{Kx^3}{3}\right]_1^3 = 1 \Rightarrow \frac{K}{3}\left[27-1\right] = 1$$

$$\boxed{K = \frac{3}{26}}$$

$$P(1.5 < x < 2.5) = \int_{1.5}^{2.5} f(x)\, dx$$

$$= \int_{1.5}^{2.5} \frac{3}{26} x^2\, dx$$

$$= \frac{\cancel{3}}{26}\left[\frac{x^3}{\cancel{3}}\right]_{1.5}^{2.5} = 0.4712$$

*** Dec 2023

(8) A R.V x has density function

$$P(x) = \begin{cases} Kx^2, & -3 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Evaluate K & find (i) $P(1 \le x \le 2)$ (ii) $P(x \le 2)$ (iii) $P(x > 1)$

Soln:-

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-3}^{3} Kx^2\,dx = 1 \Rightarrow K\frac{x^3}{3}\Big]_{-3}^{3} = 1$$

$$K[27 + 27] = 3 \Rightarrow \boxed{K = \frac{1}{18}}$$

(i) $P(1 \le x \le 2) = \int_{1}^{2} \frac{x^2}{18}\,dx = \frac{7}{54} = 0.1296$

(ii) $P(x \le 2) = \int_{-3}^{2} \frac{x^2}{18}\,dx = \frac{35}{54} = 0.6481$

(iii) $P(x > 1) = \int_{1}^{3} \frac{x^2}{18}\,dx = \frac{13}{27} = 0.4815$

---

(9) Find c.d.f for p.d.f

(i) $f(x) = \begin{cases} 6x - 6x^2, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

(ii) $f(x) = \begin{cases} \frac{x}{4} e^{-x/2}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$

(iii) Exponential distribution

Soln:- (i) $\text{c.d.f} = F(x) = \int_{-\infty}^{x} f(x)\,dx$

$$= \int_{-\infty}^{0} 0 + \int_{0}^{x} (6x - 6x^2)\,dx$$

$$= \left.\frac{6x^2}{2} - \frac{6x^3}{3}\right]_0^x = \underline{\underline{3x^2 - 2x^3}}$$

(ii) $\text{c.d.f} = F(x) = \int_{-\infty}^{x} f(x)\,dx$

$$= \int_{-\infty}^{0} 0 + \int_{0}^{x} \frac{x}{4} e^{-x/2}\,dx$$

$$= \frac{1}{4}\left[\frac{x e^{-x/2}}{-\frac{1}{2}} - (1)\frac{e^{-x/2}}{\frac{1}{4}}\right]_0^x$$

$$= \frac{1}{4}\left[-2x e^{-x/2} - 4 e^{-x/2} + 4\right]$$

$$= \underline{\underline{1 - e^{-x/2} - \frac{x}{2} e^{-x/2}}}$$

(iii) $f(x) = \begin{cases} \alpha e^{-\alpha x}, & 0 < x < \infty \\ 0, & \text{otherwise} \end{cases}$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \alpha \int_{0}^{x} e^{-\alpha x}\,dx$$

$$= \left.\alpha \frac{e^{-\alpha x}}{-\alpha}\right]_0^x = -\left[e^{-\alpha x} - 1\right]$$

$$= \underline{\underline{1 - e^{-\alpha x}}}$$

5) Find the constant K such that

$$f(x) = \begin{cases} Kx^2, & 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$ is a p.d.f.

Also compute

(i) $P(1 < x < 2)$

(ii) $P(x > 1)$

(iii) Mean

(iv) Variance

4

Soln:- W.K.T $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^3 Kx^2\,dx = 1 \implies K\frac{x^3}{3}\Big]_0^3 = 1$$

$$\frac{K}{3}(27-0) = 1 \implies \boxed{K = \frac{1}{9}}$$

(i) $P(1 < x < 2) = \int_1^2 \frac{1}{9}x^2\,dx = \frac{1}{9}\left[\frac{x^3}{3}\right]_1^2$

$$= \frac{1}{27}[8-1] = \frac{7}{27} = 0.2592$$

(ii) $P(x > 1) = \int_1^3 \frac{1}{9}x^2\,dx = \frac{1}{9}\left[\frac{x^3}{3}\right]_1^3$

$$= \frac{1}{27}[27-1] = \frac{26}{27} = 0.9630$$

(iii) Mean $= \mu = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_0^3 x \cdot \frac{1}{9} x^2\, dx$

$$= \frac{1}{9} \left. \frac{x^4}{4} \right]_0^3 = \frac{1}{36}[81-0] = \frac{9}{4} = 2.25$$

(iv) Variance $= v = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$

$$= \int_0^3 x^2 \cdot \frac{1}{9} \cdot x^2\, dx - \frac{81}{16}$$

$$= \frac{1}{9}\left[\frac{x^5}{5}\right]_0^3 - \frac{81}{16}$$

$$= \frac{1}{45}[243-0] - \frac{81}{16}$$

$$= \frac{27}{80} = 0.3375$$

*** MQP

2) The density function of a random variable X is given by,

$$f(x) = \begin{cases} K\sqrt{x}, & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

(i) Find K.

(ii) Find cdf $F(x)$ & use it to evaluate $P(0.3 < X < 0.6)$.

Soln: (i) W.K.T $\int_{-\infty}^{\infty} f(x)dx = 1$

$$K\int_0^1 x^{1/2}\,dx = 1 \Rightarrow K \left.\frac{x^{3/2}}{3/2}\right]_0^1 = 1$$

$$\frac{2K}{3}[1-0] = 1 \Rightarrow \boxed{K = \frac{3}{2}}$$

(ii) $$\text{c.d.f} = F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + K\int_0^x x^{1/2}\,dx$$

$$= K\left.\frac{x^{3/2}}{3/2}\right]_0^x = \frac{2K}{3}\left[x^{3/2} - 0\right]$$

$$= \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{2}}\left[x^{3/2}\right]$$

$$\boxed{\text{c.d.f} = x^{3/2}} = F(x)$$

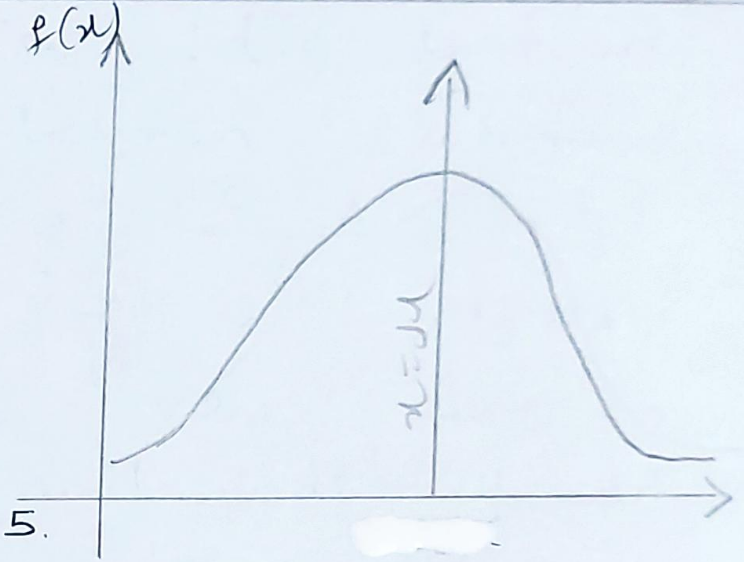
$$P(0.3 < x < 0.6) = P(0 < x < 0.6) - P(0 < x < 0.3)$$

$$= F(0.6) - F(0.3)$$

$$= 0.4648 - 0.1643$$

$$\boxed{P(0.3 < x < 0.6) = 0.3005}$$

The line $x = \mu$ divides the total area under the curve which is equal to $\underline{1}$ into two equal parts. The area to the right as to the left of line $x = \mu$ is 0.5.



$Z = \frac{x-\mu}{\sigma}$ is called standard normal variate & $F(z)$ is called standard normal curve which is symmetrical about the line $z = 0$.

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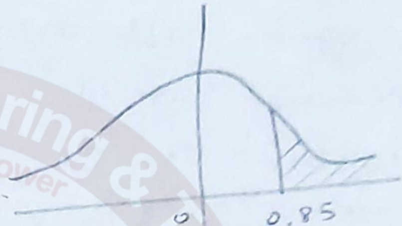
12) Evaluate using normal probability tables, (i) $P(z \geq 0.85)$
(ii) $P(-1.64 \leq z \leq -0.88)$
(iii) $P(z \leq -2.43)$
(iv) $P(|z| \leq 1.94)$

Soln:- (i) $P(z \geq 0.85) = P(z \geq 0) - P(0 < z < 0.85)$

$= 0.5 - \phi(0.85)$

$= 0.5 - 0.3023$

$= 0.1977$



(ii) $P(-1.64 \leq z \leq -0.88)$

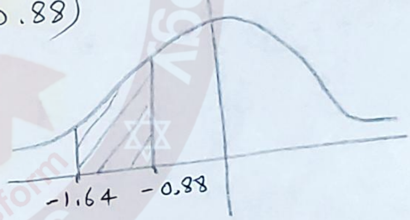
$= P(0.88 \leq z \leq 1.64)$

$= P(0 \leq z \leq 1.64) - P(0 \leq z \leq 0.88)$

$= \phi(1.64) - \phi(0.88)$

$= 0.4495 - 0.3106$

$= 0.1389$



(iii) $P(z \leq -2.43)$

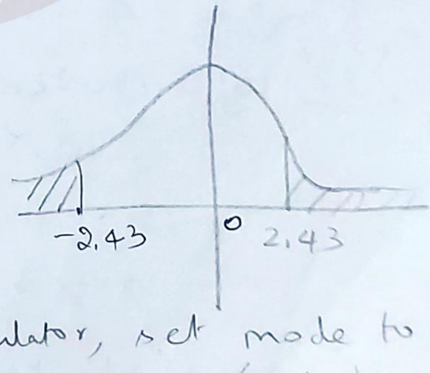
$= P(z \geq 2.43)$

$= P(z \geq 0) - P(z \leq 2.43)$

$= 0.5 - \phi(2.43)$

$= 0.5 - 0.4925$

$= 0.0075$



In calculator, set mode to SD
Then use shift (Distr) & enter
Q value as 0.85 $\longrightarrow \phi(0.85) = 0.3023$

(iv) $P(|z| \leq 1.94) = P(-1.94 \leq z \leq 1.94)$

$= 2P(0 \leq z \leq 1.94)$

$= 2\phi(1.94)$

$= 2 \times 0.4738$

$= 0.9476$

*** June 2012 (7M)

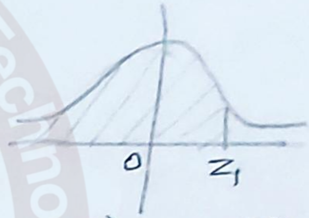
(1) In an examination 7% of students score less than 35% marks & 89% of students score less than 60% marks. Find the mean & S.D of the marks are normally distributed. It is given that $P(0 < z < 1.2263) = 0.39$ & $P(0 < z < 1.4757) = 0.43$.

Soln:- $P(x < 35) = 0.07, \quad P(x < 60) = 0.89$

Standard normal variate, $z = \dfrac{x-\mu}{\sigma}$

When $x = 35$, $z = \dfrac{35-\mu}{\sigma} = z_1$

When $x = 60$, $z = \dfrac{60-\mu}{\sigma} = z_2$



$\therefore P(z < z_1) = 0.07$

$P(z \le 0) + P(0 < z < z_1) = 0.07$

$0.5 + \phi(z_1) = 0.07$

$\phi(z_1) = -0.43$

$\phi(z_1) = -\phi(1.4757)$

$z_1 = -1.4757$

$P(z < z_2) = 0.89$

$0.5 + \phi(z_2) = 0.89$

$\phi(z_2) = 0.39$

$\phi(z_2) = \phi(1.2263)$

$z_2 = 1.2263$

$\dfrac{35-\mu}{\sigma} = -1.4757$

$\mu - 1.4757\sigma = 35 \rightarrow (1)$

$\dfrac{60-\mu}{\sigma} = 1.2263$

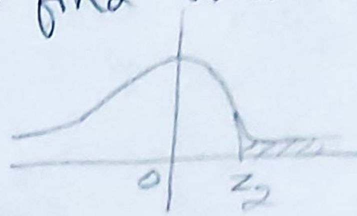
$\mu + 1.2263\sigma = 60 \rightarrow (2)$

Solving (1) & (2), $\boxed{\mu = 48.65 = \text{mean}}$

$\boxed{\sigma = 9.25 = \text{S.D}}$

(2) *** Dec 2012 (7M) (MQP)

In a normal distribution, 31% of items are under 45 & 8% are over 64. Find mean & S.D given that, $A(0.5) = 0.19$ & $A(1.4) = 0.42$ where $A(z)$ is area under standard normal curve from 0 to $z > 0$. Also find Variance



Soln:- $P(x < 45) = 0.31$

$P(x > 64) = 0.08$

$z = \dfrac{x-\mu}{\sigma}$, $z_1 = \dfrac{45-\mu}{\sigma}$ &

$z_2 = \dfrac{64-\mu}{\sigma}$

(15)

$\therefore P(z < z_1) = 0.31$ & $P(z > z_2) = 0.08$

$0.5 + \phi(z_1) = 0.31$ $\qquad$ $0.5 - \phi(z_2) = 0.08$

$\phi(z_1) = -0.19$ $\qquad$ $\phi(z_2) = 0.42$

$\phi(z_1) = -\phi(0.5)$ $\qquad$ $\phi(z_2) = \phi(1.4)$

$\boxed{z_1 = -0.5}$ $\qquad$ $\boxed{z_2 = 1.4}$

$-0.5 = \dfrac{45-\mu}{\sigma}$ & $1.4 = \dfrac{64-\mu}{\sigma}$

$\mu - 0.5\sigma = 45$ & $\mu + 1.4\sigma = 64$

Solving, $\boxed{\mu = 50}$ & $\boxed{\sigma = 10}$

Variance $= \sigma^2 = 100$

$\boxed{V = 100}$

*** June 2013 (7M) (MQP) / Dec 2023

(3) In a certain town, the duration of shower is exponentially distributed has mean 5 mins. What is the probability that shower will last for

(i) 10 mins or more

(ii) < 10 mins

(iii) b/w 10 & 12 mins.

11

Soln:- $f(x) = \alpha e^{-\alpha x}, \ x > 0$

$$\text{Mean} = \frac{1}{\alpha} = 5 \Rightarrow \alpha = \frac{1}{5}$$

$$f(x) = \frac{1}{5} e^{-x/5}$$

(i) $P(x \geq 10) = \frac{1}{5} \int_{10}^{\infty} e^{-x/5} dx$

$$= \frac{1}{\cancel{5}} \left. \frac{e^{-x/5}}{-1/\cancel{5}} \right]_{10}^{\infty} = -\left[0 - e^{-2}\right] = 0.1353$$

(ii) $P(x < 10) = \frac{1}{5} \int_{0}^{10} e^{-x/5} dx = \frac{1}{\cancel{5}} \left. \frac{e^{-x/5}}{-1/\cancel{5}} \right]_{0}^{10}$

$$= -\left[e^{-2} - 1\right] = 0.8647$$

(iii) $P(10 < x < 12) = \frac{1}{5} \int_{10}^{12} e^{-x/5} dx$

$$= \frac{1}{\cancel{5}} \left. \frac{e^{-x/5}}{-1/\cancel{5}} \right]_{10}^{12} = -\left(e^{-12/5} - e^{-2}\right)$$

$$= 0.0446$$

(4) *** Dec 2023 (7M)

The life of an electric bulb is normally distributed with average life of 2000 hrs & S.D of 60 hrs. Out of 2500 bulbs, find the no of bulbs that are likely to last (i) between 1900 & 2100 hrs. [Given $P(0 < Z \le 1.67) = 0.4525$ & $\phi(0.83) = 0.2967$]

(ii) More than 2100 hrs

(iii) Less than 1950 hrs

Soln:- $\mu = 2000, \quad \sigma = 60$

$$Z = \frac{x-\mu}{\sigma} = \frac{x-2000}{60}$$

(i) $P(1900 < x < 2100) =$

$$P\left[\frac{1900-2000}{60} < Z < \frac{2100-2000}{60}\right]$$

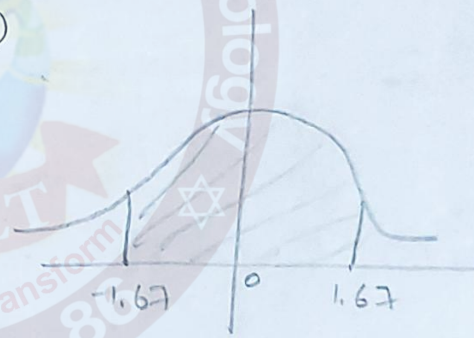
$$= P(-1.67 < Z < 1.67)$$

$$= 2P(0 < Z < 1.67)$$

$$= 2\,\phi(1.67)$$

$$= 2 \times 0.4525$$

$$= 0.905$$

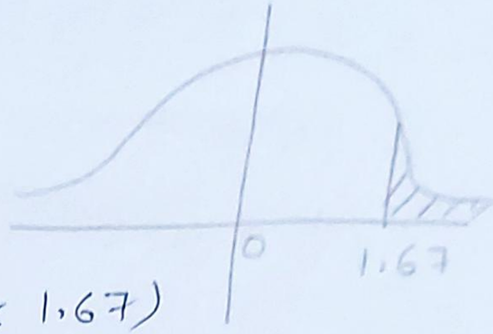


$\therefore$ No of bulbs $= 0.905 \times 2500$

$= 2263$ bulbs

(ii) $P(x > 2100)$

$= P\left(z > \frac{2100 - 2000}{60}\right)$



$= P(z > 1.67)$

$= P(z \geq 0) - P(0 \leq z \leq 1.67)$

$= 0.5 - \phi(1.67)$

$= 0.5 - 0.4525$

$= 0.0475$

$\therefore$ No of bulbs that are likely to last for more than 2100 hours $= 0.0475 \times 2500$

$= 118.8$

$\simeq 119$ bulbs

(iii) $P(x < 1950)$

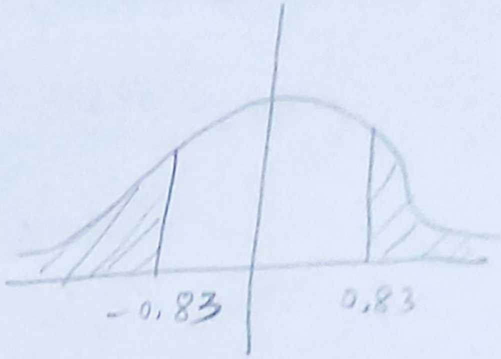
$= P\left(z < \frac{1950 - 2000}{60}\right)$

$= P(Z < -0.83)$

$= P(Z > 0.83)$

$= P(Z \geq 0) -$

$P(0 \leq Z \leq 0.83)$



$= 0.5 - \phi(0.83)$

$= 0.5 - 0.2967$

$= 0.2033$

$\therefore$ No. of bulbs that are likely to last less than 1950 hrs $= 0.2033 \times 2500$

$= 508.2$

$\simeq 508$ bulbs

F 20 B

17) If x is a normal variate with mean 30 & standard deviation 5. Find the probabilities that

(i) $26 \le x \le 40$ (ii) $x \ge 45$, given that

$\phi(0.8) = 0.2881$, $\phi(2) = 0.4772$ $\phi(3) = 0.4987$

$\phi(1) = 0.34$ 17

Soln:- standard normal variate, $z = \dfrac{x-\mu}{\sigma}$

$$z = \frac{x-30}{5}$$

(i) $P(26 \le x \le 40)$

$$= P\left(\frac{26-30}{5} \le z \le \frac{40-30}{5}\right) = P(-0.8 \le z \le 2)$$

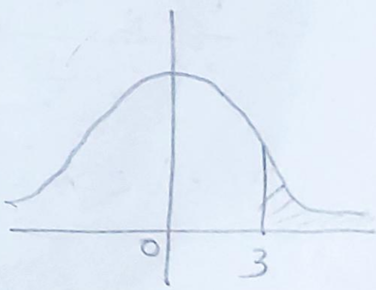
$$= P(-0.8 \le z \le 0) + P(0 \le z \le 2)$$

$$= P(0 \le z \le 0.8) + P(0 \le z \le 2)$$

$$= \phi(0.8) + \phi(2) = 0.2881 + 0.4772$$

$$= \underline{\underline{0.7653}}$$

(ii) $P(x \ge 45)$



$$= P\left(z \ge \frac{45-30}{5}\right) = P(z \ge 3)$$

$$= P(z \ge 0) - P(0 \le z \le 3)$$

$$= 0.5 - \phi(3)$$

$$= 0.5 - 0.4987$$

$$= \underline{\underline{0.0013}}$$

15) The length of telephone conversation in a booth has been an exponential distribution & found on an average to be 5 minutes. Find the probability that a random call made from this booth (i) ends less than 5 minutes

(ii) ends b/w 5 & 10 minutes

Soln:- $f(x) = \begin{cases} \alpha e^{-\alpha x}, & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases}$

14 & 21

$$\text{Mean} = \frac{1}{\alpha} = 5 \implies \alpha = \frac{1}{5}$$

$$f(x) = \begin{cases} \frac{1}{5} e^{-\frac{1}{5}x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

(i) $$P(x < 5) = \int_0^5 f(x)\,dx = \frac{1}{5}\int_0^5 e^{-\frac{1}{5}x}\,dx$$

$$= \frac{1}{\cancel{5}} \left. \frac{e^{-\frac{1}{5}x}}{-\frac{1}{\cancel{5}}} \right]_0^5 = -\left[e^{-1} - 1\right] = 0.6321$$

(ii) $$P(5 < x < 10) = \int_5^{10} f(x)\,dx = \frac{1}{5}\int_5^{10} e^{-\frac{1}{5}x}\,dx$$

$$= \frac{1}{\cancel{5}} \left\{ \frac{e^{-\frac{1}{5}x}}{-\frac{1}{\cancel{5}}} \right\}_5^{10} = -\left[e^{-2} - e^{-1}\right] = 0.2325$$

10) If x is an exponential variate with mean 3. Find (i) $P(x>1)$ (ii) $P(x<3)$

Soln:

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Mean} = \frac{1}{\alpha} = 3 \implies \alpha = \frac{1}{3}$$

$$\therefore f(x) = \begin{cases} \frac{1}{3} e^{-x/3}, & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

(i) $P(x>1) = 1 - P(x \le 1)$

$$= 1 - \frac{1}{3}\int_0^1 e^{-x/3}\,dx$$

$$= 1 - \frac{1}{3}\left[\frac{e^{-x/3}}{-1/3}\right]_0^1$$

$$= 1 + \left[e^{-1/3} - 1\right]$$

$$= 0.7165$$

OR

$$\frac{1}{3}\int_1^\infty e^{-x/3} \quad \left[-e^{-x/3}\right]_1^\infty \quad 0 - e^{-1/3} = 0.7165$$

(ii) $P(x<3) = \frac{1}{3}\int_0^3 e^{-x/3}\,dx$

$$= \frac{1}{3}\left[\frac{e^{-x/3}}{-1/3}\right]_0^3$$

$$= -\left[e^{-1} - 1\right] = 0.6321$$

*** July 2008

(11) The sales per day in a shop is exponentially distributed with the average sale amounting to Rs.100 & net profit is 8%. Find probability that the net profit exceeds Rs. 30 on 2 consecutive days.

Soln:- $\text{Mean} = \frac{1}{\alpha} = 100 \Rightarrow \alpha = 0.01$

$f(x) = \alpha e^{-\alpha x} = 0.01\, e^{-0.01x}$

Let A be the amount for which profit is 8%.

$\Rightarrow \frac{8}{100} \times A = 30 \Longrightarrow A = 375$

$P(\text{Profit} > 30) = 1 - P(\text{Profit} \le 30)$

$= 1 - P(\text{Sales} \le 375)$

$= 1 - \int_0^{375} 0.01\, e^{-0.01x}$

$= 1 + e^{-0.01x}\Big]_0^{375}$

$= \cancel{1} + e^{-3.75} - \cancel{1}$

$P(\text{Sales} > 375)$

$= \int_{375}^{\infty} 0.01\, e^{-0.01x}$

$= 0.01\left[\frac{e^{-0.01x}}{-0.01}\right]$

$= -\left[0 - e^{-3.75}\right]$

$P(\text{Profit} > 30) = e^{-3.75}$, for single day

$\therefore$ For 2 days, $P(\text{Profit} > 30) = e^{-3.75} \times e^{-3.75} = \underline{\underline{0.00055}}$

*** MQP

3) In a test on 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life 2040 hours & S.D of 60 hours. Estimate the no of bulbs likely to burn for

(8)

(i) More than 2150 hours

(ii) Less than 1950 hours

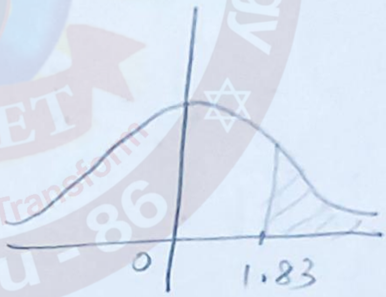
(iii) In b/w 1920 & 2160 hours

Soln:- $\mu = 2040, \quad \sigma = 60$

$$z = \frac{x-\mu}{\sigma} = \frac{x-2040}{60}$$

(i) $P(x > 2150)$

$$= P\left(z > \frac{2150-2040}{60}\right)$$



$$= P(z > 1.83)$$

$$= P(z > 0) - P(0 < z < 1.83)$$

$$= 0.5 - \phi(1.83)$$

$$= 0.5 - 0.4664 = 0.0336$$

$\therefore$ No of bulbs $= 0.0336 \times 2000$

$= 67.2 \simeq 67$ bulbs

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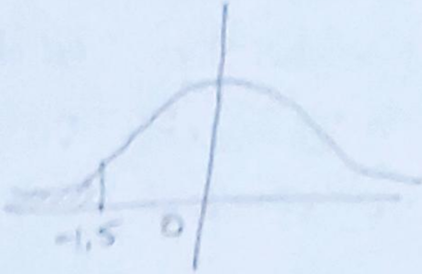
(ii) $P(x < 1950)$

$= P\left(z < \frac{1950-2040}{60}\right)$



$= P(z < -1.5)$

$= 0.5 - P(-1.5 < z < 0)$

$= 0.5 - P(0 < z < 1.5)$

$= 0.5 - \phi(1.5)$

$= 0.5 - 0.4332 = 0.0668$

$\therefore$ No of bulbs $= 0.0668 \times 2000$

$= 133.6 \simeq 134$ bulbs

(iii) $P(1920 < x < 2160)$

$= P(-2 < z < 2)$

$= 2 \times P(0 < z < 2)$

$= 2 \times \phi(2)$

$= 2 \times 0.4772 = 0.9544$

$\therefore$ No of bulbs $= 0.9544 \times 2000$

$= 1908.8 \simeq 1909$ bulbs

*** MQP

4) If the mileage (in thousands of miles) of a certain radial tyre is a R.V with exponential distribution with

mean 40,000 miles. Determine the probability that the tyre will last

(i) atleast 20,000 miles

(ii) Atmost 30,000 miles

Soln: $f(x) = \begin{cases} \alpha e^{-\alpha x}, & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases}$

$$\text{Mean} = \frac{1}{\alpha} = 40,000 \Rightarrow \boxed{\alpha = \frac{1}{40,000}}$$

$$f(x) = \begin{cases} \frac{1}{40,000} e^{-\frac{1}{40,000}x}, & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

(i) $$P(x \geq 20,000) = \int_{20,000}^{\infty} \frac{1}{40,000} e^{-\frac{1}{40,000}x} dx$$

$$= \frac{1}{\cancel{40,000}} \cdot \frac{e^{-\frac{1}{40,000}x}}{-\frac{1}{\cancel{40,000}}}\Bigg]_{20,000}^{\infty} = -\left[0 - e^{-\frac{1}{2}}\right] = e^{-\frac{1}{2}}$$

$$= 0.6065$$

(ii) $$P(x \leq 30,000) = \int_{0}^{30,000} \frac{1}{40,000} e^{-\frac{1}{40000}x} dx$$

$$= \frac{1}{\cancel{40,000}} \frac{e^{-\frac{1}{40,000}x}}{-\frac{1}{\cancel{40,000}}}\Bigg]_{0}^{30,000} = -\left[e^{-\frac{3}{4}} - 1\right]$$

$$= 0.5276$$

(13) The marks of 1000 students in an exam follows normal distribution with mean 70 & S.D '5'. Find no of students whose marks will be (i) < 65 (ii) > 75 (iii) b/w 65 & 75.

Sol'n: Let 'x' denote marks of students

$\mu = 70, \quad \sigma = 5, \quad z = \dfrac{x-70}{5}$

(22)

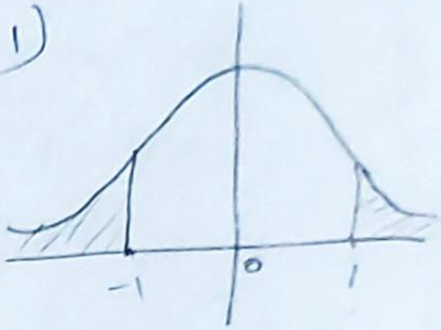
(i) $P(x < 65) = P(z < -1) = P(z > 1)$

$= P(z \geq 0) - P(0 < z < 1)$

$= 0.5 - \phi(1)$

$= 0.5 - 0.3413$

$= 0.1587$



no of students scoring < 65 = $0.1587 \times 1000 = 159$

(ii) $P(x > 75) = P(z > 1)$

$= P(z \geq 0) - P(0 < z < 1) = 0.1587$

no of students scoring > 75 = 0.1587×1000

$= 159$

(iii) $P(65 < x < 75) = P(-1 < z < 1)$

$= 2P(0 < z < 1)$

$= 2 \times 0.3413$

$= 0.6826$

$\therefore$ no. of students scoring b/w 65 & 75

$= 0.6826 \times 1000 = \underline{683}$

---

(14) For the following normal distribution find 'c' & also mean & S.D of frequency distribution, $f(x) = c \cdot e^{-\frac{1}{24}(x^2 - 6x + 4)}$

$x^2 - 6x + 4$

$= x^2 - 2(3)x + 9 - 9 + 4$

$= (x-3)^2 - 5$

Soln:-

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \longrightarrow (1)$$

Given,

$$f(x) = c\, e^{-\frac{1}{24}(x^2 - 6x + 4)}$$

$$= c\, e^{-\frac{1}{24}[(x-3)^2 - 5]}$$

$$= c\, e^{\frac{5}{24}} \cdot e^{-\frac{(x-3)^2}{24}}$$

$$f(x) = c\, e^{5/24}\, e^{-\frac{(x-3)^2}{2(\sqrt{12})^2}} \longrightarrow (2)$$

Comparing (1) & (2), $\boxed{\mu = 3}$ $\boxed{\sigma = \sqrt{12}}$

$$\frac{1}{\sigma\sqrt{2\pi}} = \frac{1}{\sqrt{12}\,\sqrt{2\pi}} = c\, e^{5/24}$$

$$\boxed{c = \frac{e^{-5/24}}{\sqrt{24\pi}}}$$

(16) The mean weight of 500 students during a medical exam was found to be 50 kgs & S.D weight 6 kgs. Assuming that the weights are normally distributed, find no of students having weight

(i) b/w 40 & 50 kgs

(ii) more than 60 kgs given that $\phi(1.67) = 0.4525$, where $\phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2}\, dz$

Soln:- $\mu = 50$, $\sigma = 6$, $Z_1 = \frac{x-\mu}{\sigma}$

(i) $P(40 \le x \le 50)$

$Z_1 = \frac{40-50}{6} = -1.67$

$= P(-1.67 \le z \le 0)$

$Z_2 = \frac{50-50}{6} = 0$

$= P(0 \le Z \le 1.67)$

$= \phi(1.67)$

$= 0.4525$

$\therefore$ no of students having weight b/w 40 & 50 kgs $= 0.4525 \times 500$

$= 226$ students

(ii) $P(x > 60)$

$Z = \frac{60-50}{6} = 1.67$

$= P(Z > 1.67)$

$= P(Z \ge 0) - P(0 \le Z \le 1.67)$

$= \frac{1}{2} - \phi(1.67)$

$= 0.5 - 0.4525$

$= 0.0475$

$\therefore$ no of students having weight more than 60 kgs $= 0.0475 \times 500$

$= 24$ students

Standard Normal Distribution Table

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998

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