BGS COLLEGE OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF PHYSICS

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MODULE - 01

Lasers & Optical Fibers

By

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Department of Physics
Module – 1
Lasers & Optical Fibers

Syllabus:

Laser and Optical Fibers (8 Hrs):

LASER: Basic properties of a LASER beam, Interaction of Radiation with Matter, Einstein's A and BCoefficients, Laser Action, Population Inversion, Metastable State, Requisites of a laser system, Semiconductor Diode Laser, Applications: Bar code scanner, Laser Printer, Laser Cooling (Qualitative), Numerical Problems.

Optical Fiber: Principle and structure, Propagation Mechanism, Acceptance angle and Numerical Aperture (NA) and derivation ofExpression for NA, Classification of Optical Fibers, Attenuation and Fiber Losses, Applications: Fiber Optic networking, Fiber Optic Communication.

Pre requisite: Properties of light

Self-learning: Total Internal Reflection (Optical Fibers)

LASER

The word Laser stands for Light Amplification by Stimulated Emission of Radiation and it is a device that amplifies light.

Properties of Laser

The laser light exhibits some peculiar properties compared with the conventional light which make it unique. i.e.

- Monochromatic
- Coherence
- Directionality
- Highly Intense or Brightness

1. Monochromatic: Monochromatic light is a light containing a single colour or wavelength. Thelights emitted from ordinary light sources have different energies, frequencies, wavelengths,

or colors. But laser light has a single wavelength or colour. Laser light covers a very narrow range offrequencies or wavelengths. This can be due to the stimulated characteristics of laser light. The bandwidth of the conventional monochromatic light source is 1000 Å. But the bandwidth of an ordinary light source is 10 Å. For a highly sensitive laser source it is 10-8 Å.

2. Coherence: A predictable correlation of the amplitude and phase at any one point with another point is called coherence. That means if two or more waves of same frequency are in the same phase or have constant phase difference then these waves are said to be coherent in nature.

In the case of conventional light, the property of coherence exhibits between a source and its virtual source whereas in the case of laser the property coherence exists between any two or more light waves. There are two types of coherence. Temporal coherence and Spatial coherence.

3. Directionality: The light ray coming from an ordinary light source travels in all directions, but laser light travels in a single direction.

4. Highly Intense or Brightness: Laser light is highly intense than conventional light. We know that the intensity of a wave is the energy per unit time flowing through a specific area. A one millwatt He-Ne laser is more intense than the sun intensity. This is because of the coherence and directionality of the laser.

Interaction of radiation with matter

The interaction of light with matter makes the transition of an atom or a molecule from one energy state to another. If the transition is from a lower state to a higher state it absorbs the incident energy. If the transition is from a higher state to a lower state, the atom emits a difference in energy levels between two orbits in the form of electromagnetic radiation.

If ΔE is the difference between the two energy levels, then, according to Max Planck, $\Delta E = h\nu = E_2 - E_1$

$$\nu = \frac{E_2 - E_1}{h}$$

Emission or Absorption takes by the quantum of energy called photons. The energy of one photon is hν which is called quantum energy or one photon energy, h = Planck's constant=6.625×10^{-34} JS, and 'ν' is the frequency of radiation emitted or absorbed.

The possible ways of interactions are

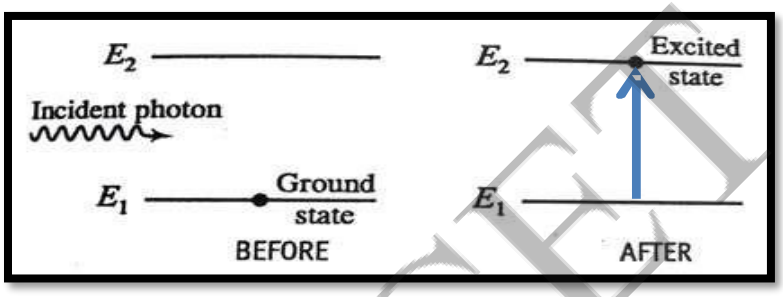
1) Induced absorption
2) Spontaneous emission
3) Stimulated emission

1) Induced absorption

"The process in which an atom is raised from a lower energy state to a higher state, due toabsorption of an incident photon by an atom is called induced absorption".

Explanation: Let 'E_1' and 'E_2' be the energies of two energy levels in a material, such that $E_2>E_1$. When a photon of energy **hν** is incident on an atom at level E_1, the atom is stimulated or induced to go to a higher energy level by absorbing the energy. The process is represented as

Atom + Photon → Atom*



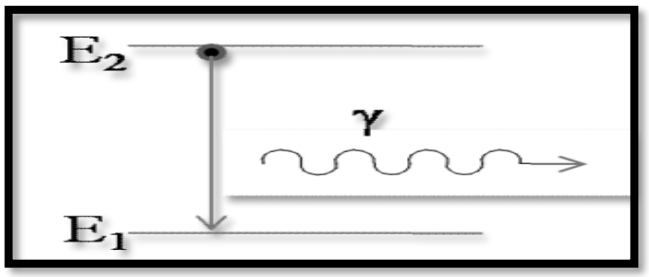
2) Spontaneous Emission:

The emission of a photon by an atom of a system due to which an "atom makes the transition from a higher energy state to a lower energy state by itself without the aid of any external energy" is called spontaneous emission.

Explanation: Let 'E_1' and 'E_2' be two energy levels in a material, such that $E_2>E_1$. The time of stay of an atom in the excited state is usually very short of the order of **10^{-8}** and it is called the lifetime of an atom. After this time the atom undergoes a transition to a lower energy level without the aid of external energy. This process is called Spontaneous Emission. This process is represented as

Atom* → Atom + Photon

The photons emitted in spontaneous emission may not have the same direction and phase. Therefore emitted light is incoherent light.



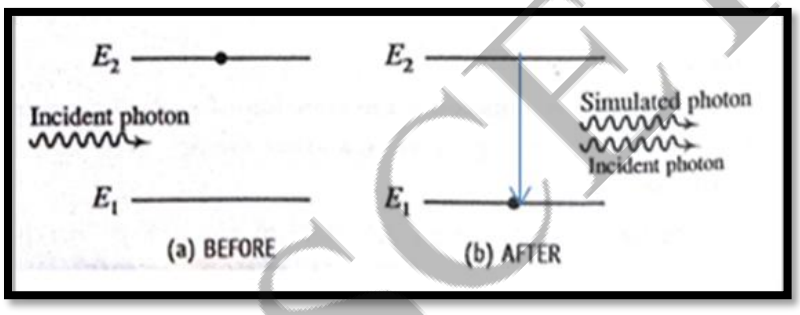
3) Stimulated Emission:

Emission of a photon by an atom when a photon of suitable energy incident on the atom, due to which the atom makes the transition from a higher energy state to a lower energy state is called Stimulated emission.

Explanation:

Let 'E_1' and 'E_2' be two energy levels in a material, such that $E_2 > E_1$. Let the atom is at a higher level of E_2. Let the energy of the incident photon be equal **hν** which is equal to the difference of the energy levels between two energy levels. The atom absorbing energy; forces the atom to get de-excited from higher level E_2 to lower level E_1.

$h\nu = E_2 - E_1$ is the energy of the photon emitted.



The Incident photon stimulates or induces the excited atom to emit a photon of exactly the same energy as that of the incident photons. In this process, two photons are emitted. The photons thus emitted are called stimulated photons which are identical in all respect i.e. having the same phase, same frequency, and direction with respect to the incident photon. Thus stimulated emission is the particular type of emission which is responsible for the emission of laser light.

The process is represented by,

Atom* + Photon → Atom + 2 photons

Consider two energy level quantum systems with E_1 be the energy of the lowest energy state and E_2 be the energy corresponds to higher energy state. Let N_1 band N_2 be the number of atoms per unit volume of the system in energy states E_1 and E_2 respectively. Let us consider a stream of photons having the wavelength range λ to $\lambda + d\lambda$ be incident on the system. Then the energy incident per unit volume of the system in the frequency interval of λ and $\lambda + d\lambda$ is given by $E_\nu\, du$.

i) **Induced Absorption**: In this process, the electron/atom makes transition from lower energy state E_1 to higher energy state E_2, when it absorbs a photon of energy $\Delta E = E_2 - E_1$ incident on the system. The number of such absorptions per unit time, per unit volume is called rate of induced absorption.

The rate induced absorption depends on the two factors,

a) Number of atoms per unit volume in lower energy state i.e., N_1 and

b) Incident energy density E_u.

Therefore, Rate of induced absorption $\alpha\, N_1 E_\nu$

Rate of induced absorption $= B_{12} N_1 E\nu$

Where, B_{12} is the proportionality constant called Einstein's coefficient of induced absorption.

ii) **Spontaneous emission**: The spontaneous emission occurs when an electron/atom undergoestransition from higher energy state E_2 to lower energy state E_1, by emitting a photon of energy $\Delta E = E_2 - E_1$. But this emission is without being aided by any external agency. Hence, it is independent of the incident energy density E_u and depends only on the number of atoms in the higher energy state N_2. The number of such emissions per unit time, per unitvolume is called the rate of spontaneous emission.

Then, the rate of spontaneous emission $\alpha\, N_2$

The rate of spontaneous emission $= A_{21}\, N_2$

Where, A_{21} is the proportionality constant called Einstein's coefficient of spontaneous emission.

iii) **The Stimulated emission**: Stimulated emission requires external photon of right energy $\Delta E = E_2 - E_1$ to stimulate the electron/atom to the downward transition, results in the emission of stimulated photon. Hence, the emitted photons are depending on the incident energy density E_u, and depends on the number of atoms in the higher energy state.

Then, the rate of Stimulated emission $\alpha\ N_2\ E_\nu$

Then the rate of Stimulated emission $= B_{21} N_2\, E_\nu$

Where, B_{21} is the proportionality constant calledEinstein's coefficient of stimulated emission.

Consider the system under thermal equilibrium, under such conditions the number of such photons absorbed per second is equal to the number photos emitted per second both by the spontaneous emission and by stimulated emission processes.

∴ Under thermal equilibrium,

Rate of Absorption = Rate of spontaneous emission + Rate of stimulated emission

$$B_{12} N_1 E_\nu = \mathrm{A}_{21}\,\mathrm{N}_2 + B_{21} N_2 E_\nu$$

$$E_\nu\,(B_{12} N_1 - B_{21} N_2) = A_{21}\,\mathrm{N}_2$$

$$E_\nu = \frac{A_{21}N_2}{B_{12}N_1 - B_{21}N_2}$$

$$E_\nu = \frac{A_{21}}{B_{21}}\left[\frac{1}{\frac{B_{12}N_1}{B_{21}N_2}-1}\right] \text{---------- (1)}$$

We have the ratio of population by Boltzmann's law

$$\frac{N_1}{N_2} = e^{\frac{h\nu}{kt}} \text{ ---------- (2)}$$

Substituting Equation (2) in (1), we get,

$$E_\nu = \frac{A_{21}}{B_{21}}\left[\frac{1}{\frac{B_{12}}{B_{21}}e^{\frac{h\nu}{kt}}-1}\right] \text{ ---------- (3)}$$

According to Planck's radiation law,

$$E_\nu = \frac{8\pi h\nu^3}{c^3}\left[\frac{1}{e^{\frac{h\nu}{kt}}-1}\right] \text{ ----------(4)}$$

On comparison of equations (3) and (4) we get

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{c^3} \qquad \frac{B_{12}}{B_{21}} = 1 \text{ or } B_{12} = B_{21}$$

This means that the probability of induced absorption is equal to the probability of stimulated emission.Because of the above identity, the subscripts could be dropped, and hence A_{21}=A and B_{21}=B.

Therefore, the expression for energy density, in terms of Einstein's coefficients can be expressed as,

$$\boldsymbol{E_\nu = \frac{A}{B}\left[\frac{1}{e^{\frac{h\nu}{kt}}-1}\right]}$$

Condition for Laser Action:

Continuous emission of laser light without stopping is called laser action. The required conditions for laser action are Metastable states and Population Inversion.

Metastable state:

- Population inversion is possible to achieve in certain systems which possess a special kind of excited states called metastable states.

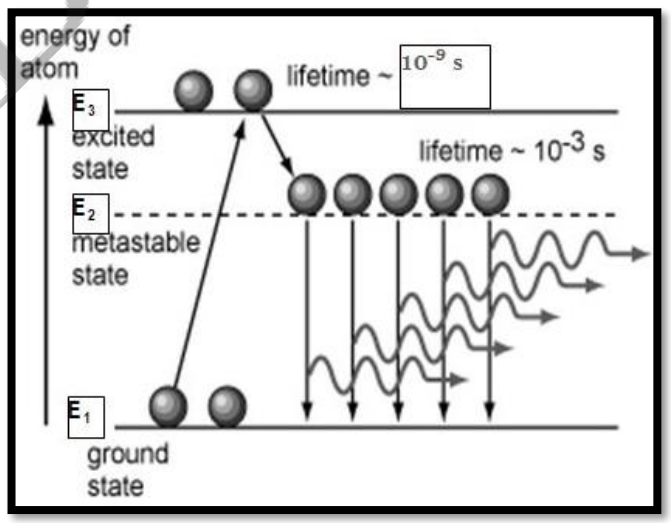
- '***Metastable state'*** is an intermediate energy state present between an ordinary excited state and the ground state where the atoms stay for a longer duration of the order of **10^{-3}** S.
- On supplying energy to the system; it is possible to achieve population inversion in a metastable state.
- The process of exciting atoms from a lower energy level to a higher energy level is called pumping and the source of energy required for this purpose is called the pumping source or excitation source.
- If an atom is made to go to one of its excited states by pumping, it stays in the ordinary excited state for 10^{-8} S, and then it comes to a metastable state by the nonradioactive transition. At metastable state, atoms stay for a longer time of 10^{-3} S. Thus, if the atoms are continuously raised to a metastable state, their population becomes more than the population of its lower energy state. This is a required condition for laser action.

Population Inversion:

According to **Boltzmann's factor [** $\frac{N_2}{N_1} = e^{\frac{h\nu}{kt}}$ **],** the number of atoms present in the ordinary excited state is less than that of in the lower energy state

The reason is that the average lifetime of atoms in an ordinary excited state is of the order of **10^{-8} S**. After this time, all the atoms undergo spontaneous emission to the ground state. Hence laser action is not possible. For laser action, an inverted condition of the above, called population inversion is required i.e $N_2 > N_1$.

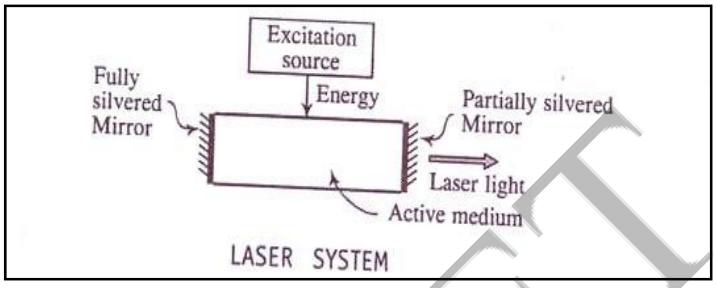
The state in which there exist a large number of atoms at higher energy levels than lower energy levels iscalled population inversion.



Requisites of a Laser system

The important requisites for a laser system are

1) An Active medium /Lasing medium
2) An Excitation source for pumping system
3) A Resonant cavity / an optical cavity



Active medium:

A medium consisting of metastable states, which supports population inversion is called an active medium, Such a medium consists of three or four energy levels. Accordingly, the laser system is named a three-level laser system or a four-level laser system.

In the active medium, stimulated emissions are initiated by a few spontaneously emitted photons. Each stimulated emission gives rise to two identical photons. These two photons initiate two excited atomsto undergo stimulated emissions and there will be four identical photons. Thus photon multiplication takes place and therefore light is amplified in the medium.

Pumping:

To achieve population inversion, atoms must be continuously excited from the ground state to the higher energy state by supplying energy. The external energy used to excite atoms to achieve population inversionis called excitation energy/source. The source of energy may be optical, electrical, thermal, or chemical.

Accordingly, the pumping action is named as 'Optical pumping' (ex. Ruby laser), 'Electrical pumping'(Ex. He-Ne laser, CO_2 laser, Ga-As laser), and 'Chemical pumping'.

Resonant cavity:

The resonant cavity consists of two mirrors fixed on either side along the length of the active medium. One of the mirrors is completely silvered to reflect the entire incident light whereas the other is partially silvered to exit the laser. Therefore active medium together with a reflecting mirror is called a resonant cavity.

The cavity reflects photons into the active medium so that stimulated emission continues. Due to reflection, photons travel back & forth several times between the mirrors, photon multiplication takes place. When intensity becomes sufficiently high part of the light is emitted through semi silvered mirror.

Further, the resonator selects and amplifies only those photons which are traveling along the cavity. Photons traveling in other directions are reflected sideways and are removed. Thus resonator makes the beam unidirectional. At the mirrors, the incident wave and the reflected wave must superpose/interfere constructively to form a standing wave. For this, the cavity length must be an integral multiple of half the wavelength. That is L = $\frac{m\lambda}{2}$ where 'm' is the number of modes.

$$L = \text{integral multiple of } \frac{\lambda}{2}$$

$$L = \frac{m\lambda}{2}$$

$$\lambda = \frac{2L}{m}$$

where 'L' is the length of the laser cavity and 'λ' is the wavelength of the emitted photons

Semiconductor Laser diode: Gallium Arsenide Laser

The most compact of all the lasers is the semiconductor laser diode. This laser diode consists of a p-n junction doped in a single crystal of a suitable semiconductor such as Gallium-Arsenide.

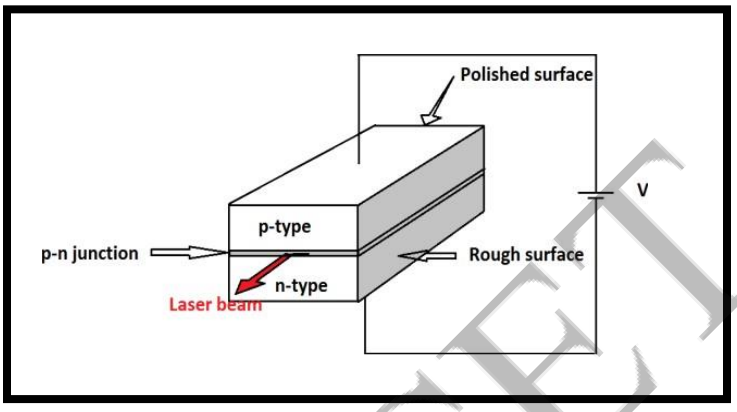
Principle: The laser transition can be possible only with the direct band gap semiconductors. Therefore, direct band gap semiconducting materials like Ga-As are used for laser action. The p-type material is formed by addition of dopant like tellurium and n-type material is formed by addition of zinc. This heavilydoped semiconducting material consists of filled energy states at the bottom of the conduction band andtop of the valance band.

When Semiconductor laser diode in the forward biased, holes are injected into p-type and electrons into n-type of the junction. The recombination of electrons and holes within the junction region results in the emission of photons. If the junction current is large enough, population inversion can be obtained between the electron levels and hole levels, that is increasing the population of energy levels near the bottom of the conduction band and top of the valance band. Hence stimulated emission can be achieved.

Construction: A schematic diagram of a Ga-As semiconductor laser is shown in the Figure.

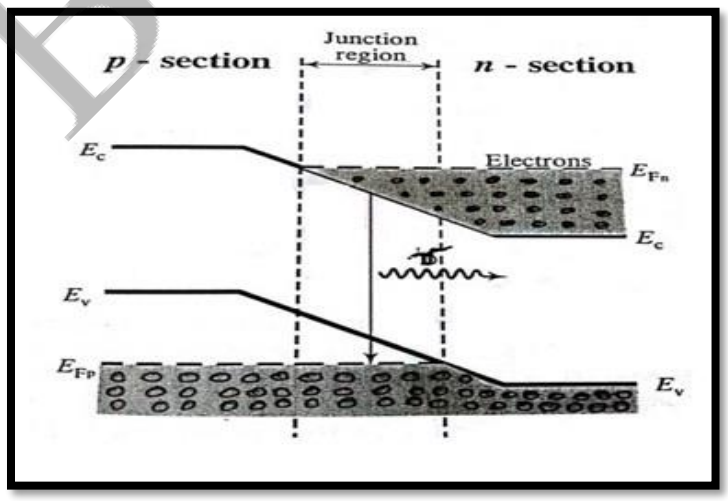
- The diode is extremely small in size with sides of the order of 1mm and junction width varying from1μm to 100 μm.

- The p-n junction acts as an active medium. The faces are made optically flat and parallel. One sideis fully polished to reflect light and the other side is partially silvered to transmit light; this forms a resonant cavity.
- The end faces of p and n junctions parallel to the plane of the junctions are provided with electrodes for biasing. These electrodes are forward-biased using a DC power supply.



Working:

The energy band diagram for the diode under forward bias is as shown. Before applying voltage, the concentration of electrons at the bottom of conduction band will be lesser than the concentration of electrons in the valence band. This is due to more number of spontaneous emissions. When a junction is forward biased, electrons and holes are injected into the junction i.e. Charge carriers are pumped by the D.C voltage source. The electrons are injected from the n-type region, and holes are injected from the p-side region into the junction. i.e. electrons in valence band (VB) are excited into conduction band (CB).



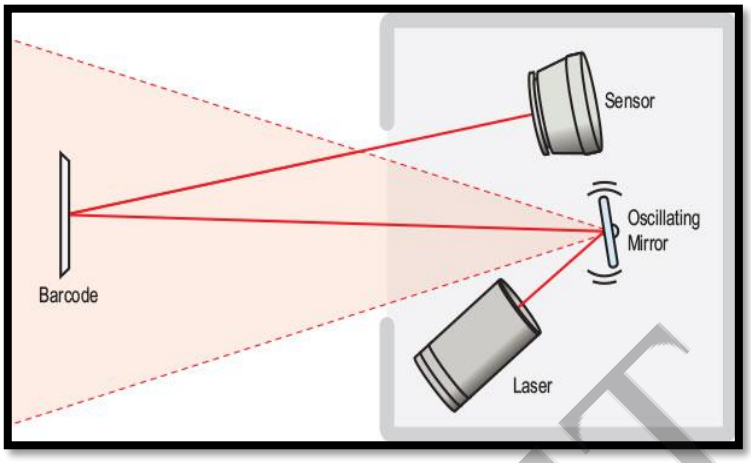
- ❖ When an electron leaves VB, a vacancy is created in VB. As the current increases, more and more electrons are excited into the CB. Therefore, at thermal equilibrium, the bottom of CB consists of fullof electrons, and the top of VB is full of holes (absence of electrons).
- ❖ The state of an excited electron in the conduction band is unstable and hence within a short duration of 10^{-8} S, the electron falls back into the valence band to recombine with a hole. The energy associated with this recombination is emitted in the form of a photon.
- ❖ This photon stimulates another electron in the conduction band to recombine with a hole in the valence band with the release of another photon.
- ❖ These photons are coherent and travel together which are get reflected from the end faces. These reflected photons will further stimulate electron-hole recombination with the release of additional photons. These photons get resonated by traveling back and forth and finally emerge from the partially reflecting face generating a continuous laser beam.
- ❖ In terms of the Energy gap and Fermi level, this can be explained as follows. If E_{Fn} and E_{Fp} are the Fermi levels in the n- region and p- region respectively. Due to the excitation of electrons from VB to CB, the Fermi level due to electrons (E_{Fn}) increases in the conduction band and that due to holes (E_{Fp}) in the valence band decreases. Thus population inversion is created within the junction. It is as shownin the diagram.
- ❖ The energy gap of the GaAs diode is 1.4 eV and hence the wavelength of emitted laser light is $\lambda = \frac{hC}{E_g} = 8400A^0$ in the IR region.

Applications of lasers:

Bar code scanner:

A barcode reader also called a price scanner or point-of-sale (POS) scanner, is a hand-held or stationaryinput device used to capture and read the information contained in a barcode.
A barcode reader comprises a scanner, a decoder (either built-in or external), and a cable used to connect the reader to the computer or POS system.
Because a barcode reader merely captures and translates the barcode into numbers and/or letters, the data must be sent to a computer so a software application can translate the data into information. Barcode scanners can be connected to a computer through a serial port, keyboard port, or interface device called a wedge. A barcode reader works by directing a beam of light across the barcode and measuring the amount and pattern of the light that is reflected.
The scanner converts the light energy into electrical energy, which is then converted into data by

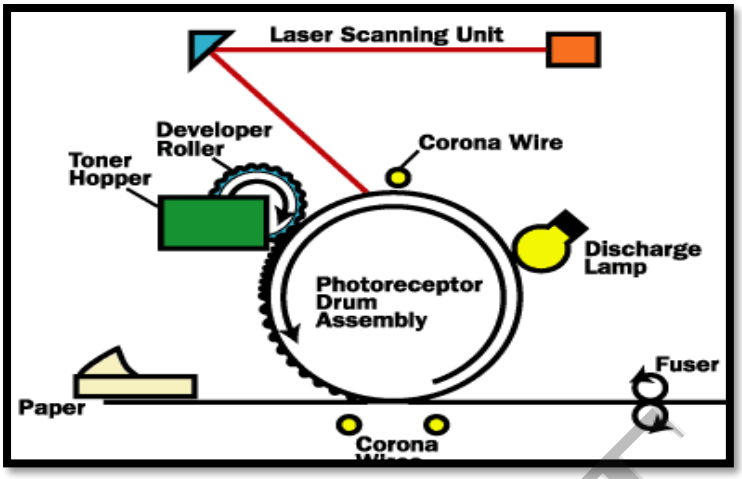
the decoder and forwarded to a computer.



Laser Printer:

The laser printing process involves following steps.

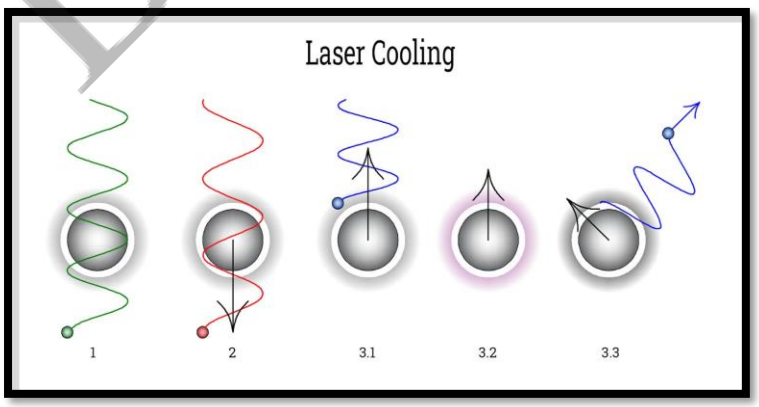
1. The data required to be printed is transferred from the computer to the laser printer. This is usually via an external cable or wirelessly if the printer has wireless capabilities.
2. The printer then has to reach the required temperature via the heating of the corona wire. This wire once heated, passes an electrical static charge to the drum unit.
3. The drum unit, now positively charged, is ready to receive the laser beam and begin the datatransfer process directly onto the drum.
4. Once the laser is activated, the beam reflects off a moving mirror unit which directs the beamdirectly onto the drum unit.
5. In the areas where the beam hits the drum, the charge is changed from negative to positive. The negatively charged areas now represent where toner particles will adhere to the drum and be directly transferred onto the paper.
6. The ink roller now begins to coat the drum with toner. Toner is comprised of microscopic inkparticles which, now positively charged, adhere to the negatively charged areas on the drum unit.
7. A positively charged sheet of paper is now passed close to the drum, attracting the negativelycharged toner particles onto the page.
8. The paper, now containing the inked content, is passed into the fuser unit where the rollers fuse thetoner particles to the paper.
9. The page is then passed through to the other side of the copier, and now have one successfulprintout.



Laser Cooling:

Laser cooling includes a number of techniques in which atoms, molecules, and small mechanical systemsare cooled, often approaching temperatures near absolute zero. Laser cooling techniques rely on the factthat when an object (usually an atom) absorbs and re-emits a photon (a particle of light)its momentum changes. For an ensemble of particles, their thermodynamic temperature is proportional to the variance in their velocity. That is, more homogeneous velocities among particles correspond to a lower temperature. Laser cooling techniques combine atomic spectroscopy with the aforementioned mechanical effect of light to compress the velocity distribution of an ensemble of particles, thereby cooling the particles.

If an atom is traveling toward a laser beam and absorbs a photon from the laser, it will be slowed by the fact that the photon has momentum $P = E/C = h/\lambda$.

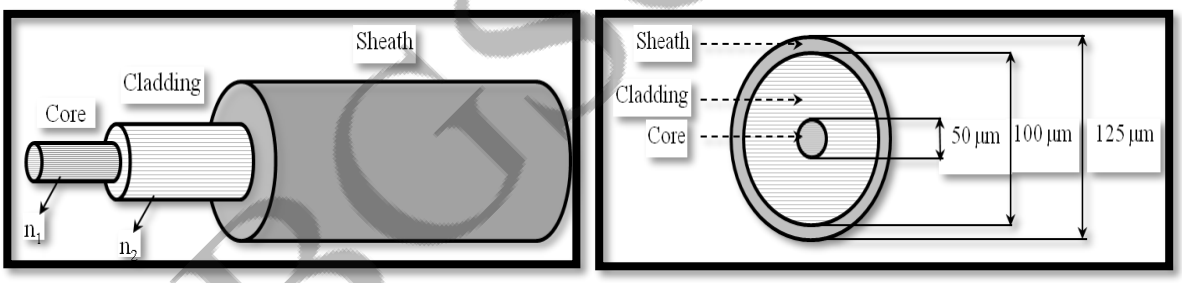


Optical Fibers

Optical fibers are used in optics as **waveguides.** They are transparent dielectric materials able to guidevisible and infrared light over long distances. The work on the principle of *Total Internal Reflection (TIR*). When a ray of light travels from a denser medium to a rarer medium and if the angle of incidence is greater than the critical angle then the light gets totally reflected.

Construction

The structure of an optical fiber is shown in Figure. The optical fiber consists of fibers of plastic or glass cylinder called **'*Core*'**. The core is surrounded by a concentric cylinder of similar material called **'*Cladding*'** but of a lower refractive index (n_2) than that of the core (n_1). The purpose of the cladding is to make the light to be confined to the core. The core and cladding are made up of either glass or plastic (dielectric). The cladding is enclosed in a jacket called **'*Sheath*'** made of polyurethane, which protects the fiber from abrasions, contaminations, and the harmful influence of moisture. In addition, it improves the mechanical strength of the fiber.



Optical fiber structure

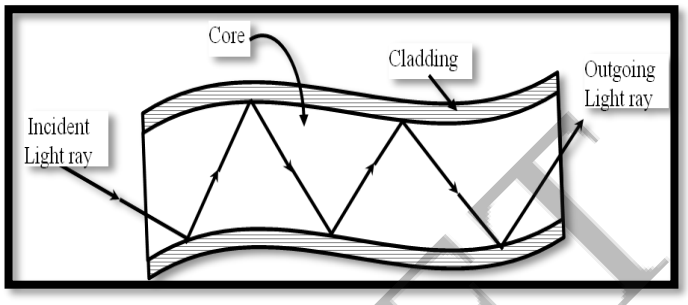
Propagation Mechanism:

Working or guiding mechanism in optical fiber as a waveguide

A tubular structure through which light energy could be guided in the form of a wave is called a **"*waveguide*".** Since light waves can be guided through the optical fiber, it is called a "*light guide*". It is also called "fiber waveguide" or "fiber light guide".

Light travels as an electromagnetic wave through an optical fiber. Light from a laser **or** some other source enters one end of the optical fiber. As the light travels through the core, it strikes the interface of the core and cladding at an angle greater than the critical angle of incidence.

The ray undergoes a series of total internal reflections. Thus, the light will be guided through the core even if the fiber is bent. Since each reflection is a total internal reflection, the light signal sustains its strength and also confines itself completely within the core during the course of propagation. At the other end of the fiber, the light is received by a detector such as a photosensitive device.



Total internal reflection

Consider a ray of light propagating from a medium of higher refractive index n_1 into medium of lower refractive index n_2 at the interface by making an angle 'i' to the normal. The refracted ray bends away from the normal by making an angle 'r' is show in the Figure 1.1.

If the angle 'i' of incident ray increases and if it becomes equal to critical angle θ_C then the refracted ray propagates through boundary between two media by making an angle 90^0 as shown in the Figure 1.2.

When the angle 'i' of incident ray exceeds the critical angle θ_C, the incident ray gets reflected back into the same medium. Hence there is no loss if intensity of refracted ray as shown in Figure 1.3. The reflection of light into the same medium for the incident angle greater than critical angle is known as total internal reflection.

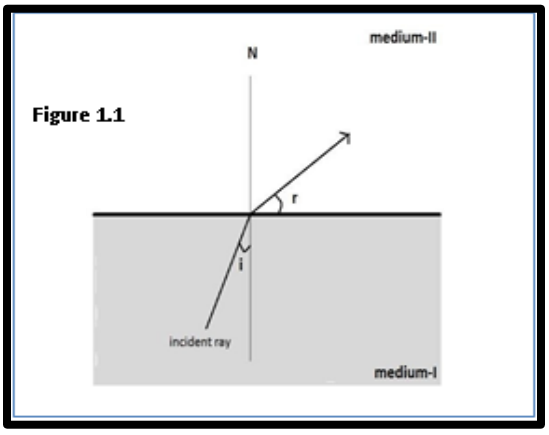


Figure 1.1

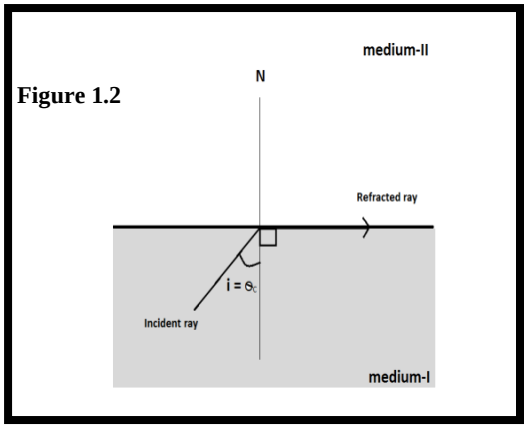


Figure 1.2

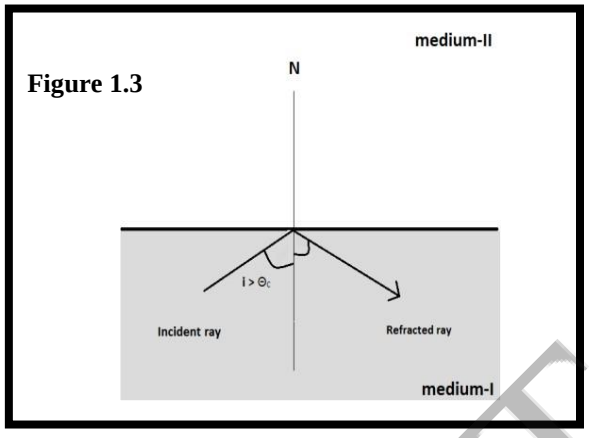


Figure 1.3

Applying the Snell's law for the incident ray shown in the Figure 1.1

$$n_1 \sin i = n_2 \sin r$$

For $i = \theta_C$ and $r = 90^0$. The above equation simplified to

$$n_1 \sin \theta_C = n_2 \sin 90^0$$

$$n_1 \sin \theta_C = n_2$$

$$\sin \theta_c = \frac{n_2}{n_1}$$

$$\theta_O = sin^{-1}\left(\frac{n_2}{n_1}\right)$$

When the angle of incidence greater than the critical angle i.e, $\theta_i > \theta_0$, then the total internal reflection takes place as shown in Figure 1.3

Define the terms

Critical propagation angle: For the ray to undergo TIR, the angle of refraction with the axis of the fiber must be less than a certain maximum value called Critical propagation angle (θ1). If the angle of refraction with the axis of the fiber is greater than this angle, the angle of incidence at the interface becomes less than the critical angle, and hence ray escapes outside the fiber without undergoing TIR.

Acceptance angle or acceptance cone half angle: For the ray to undergo successive total internal reflection, the angle made by the incident ray at the axis of the fiber must be less than a certain maximum value called acceptance angle or acceptance cone half angle.

OR

If the incident ray is rotated about the fiber axis keeping θ_o constant, it forms a cone. The rays which are incident within this cone travel inside the fiber and undergo TIR is called acceptance angle or acceptance cone half angle.

Numerical aperture: The numerical aperture of an optical fiber is a measure of light gathering capacity of the fiber, it is equal to the sine of the waveguide acceptance angle. i.e

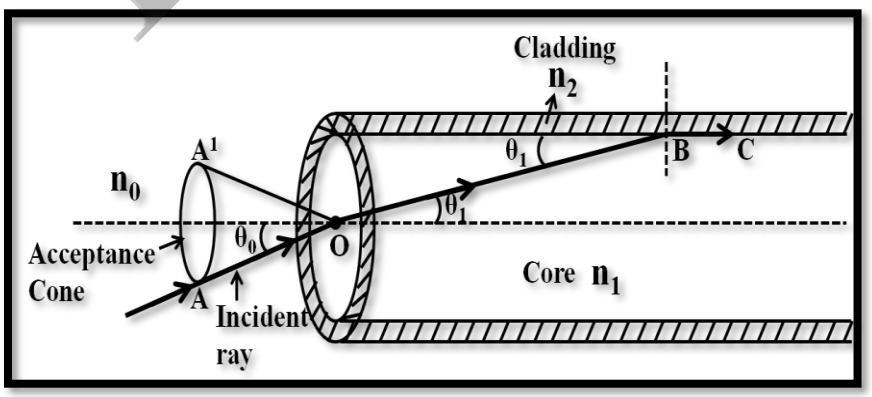
$$\sin\theta_0 = \frac{n_1}{n_0}\sqrt{\left(\frac{n_1^2 - n_1^2}{n_1^2}\right)}$$

Where, n_1, n_2 and n_0 represent refractive indices of the core, cladding, and surrounding medium respectively.

Expression for Acceptance angle and Numerical Aperture:

Consider a light ray AO incident at an angle '$\theta_i = \theta_0$' enters into the fiber. Let 'θ_1' be the angle of refraction for the ray OB. The refracted ray OB incident at a critical angle $(90^\circ - \theta_1)$ at B; grazes the interface between the core and cladding along BC. If the angle of incidence is greater than the critical angle, it undergoes total internal reflection. Thus θ_0 is called the waveguide acceptance angle and $\sin\theta_0$ is called the numerical aperture (NA).

NA of an optical fiber is the light-gathering capacity of the fiber.



The main function of optical fiber is to accept and transmit as much light as possible. The light-gathering ability of a fiber depends on two factors, viz. *core size*, and the *numerical aperture*. Let n_0, n_1, and n_2 be the refractive indices of the medium, core, and cladding respectively.From Snell's law,

$$n_0 \sin\theta_0 = n_1 \sin\theta_1$$

$$n_0 \sin\theta_0 = n_1 \sqrt{1 - cos^2\,\theta_1} \text{---------- (1)}$$

Now, Applying the Snell's law at core-cladding interface at point B

$$n_1 \sin(90 - \theta_1) = n_2 \sin 90^o$$

$$n_1 \cos\theta_1 = n_2$$

$$\cos\theta_1 = \frac{n_2}{n_1} \text{----------(2)}$$

Substituting equation (2) in (1) we get

$$n_0 sin\,\theta_0 = n_1 \sqrt{1 - \left(\frac{n_2}{n_1}\right)^2}$$

$$\sin\theta_0 = \frac{n_1}{n_0} \sqrt{\left(\frac{n_1^2 - n_2^2}{n_1^2}\right)}$$

$$\sin\theta_0 = \sqrt{\left(\frac{n_1^2 - n_2^2}{n_0^2}\right)}$$

$$\theta_0 = sin^{-1} \sqrt{\left(\frac{n_1^2 - n_2^2}{n_0^2}\right)}$$

This is an expression for acceptance angle or half cone acceptance angle for an optical fiber.

The light incident within the half cone acceptance angle, the refracted rays undergo total internal reflection in the core part of an optical fiber. The light ray can propagate through the optical fiber only when the angle of incidence is less than the acceptance angle.

$$\theta_i < \theta_0 \quad \text{or} \quad sin\theta_i < sin\theta_0 \quad \text{or} \quad sin\theta_i < NA$$

Numerical aperture (NA) is the most important parameter of an optical fiber. It is the measure of how much light can be collected by the fiber for the transmission. It is the sin of the half cone acceptance angle, and is given by.

$$NA = sin\,\theta_0 = \sqrt{\left(\frac{n_1^2 - n_2^2}{n_0^2}\right)}$$

For air medium $n_o = 1$

$$NA = sin\,\theta_0 = \sqrt{n_1^2 - n_2^2}$$

This relation shows the light gathering ability of an optical fiber increases with its numerical aperture.

Fraction index change (Δ)

The graph obtained by plotting the refractive index of the fiber with respect to radial distance from the axis of the fiber is called the refractive index profile.

It is the ratio of the refractive index difference of the core and cladding to the refractive index of core of the optical fiber.

$$\Delta = \frac{n_1 - n_2}{n_1}$$

Relation between NA and Δ

We know that, Fraction index change, $\Delta = \frac{n_1 - n_2}{n_1}$

$$n_1 - n_2 = \Delta n_1$$

Numerical aperture, $NA = \sqrt{n_1^2 - n_2^2}$

$$NA = \sqrt{(n_1 + n_2)(n_1 - n_2)}$$

$$NA = \sqrt{(n_1 + n_2)\Delta n_1)}$$

$$NA = \sqrt{2n_1 n_1\,\Delta}$$

$$NA = \sqrt{2n_1^2\Delta}$$

$$NA = n_1\sqrt{2\Delta}$$

As Δ increases, NA increases and thus enhances the light gathering capacity of the fiber.

Modes of Propagation

Propagation of each ray in an optical fiber is called as mode. Many such rays propagation is called as modes of propagation.

The number of modes supported for propagation in the fiber is determined by a parameter called V-number and is given by,

$$V = \frac{\pi d}{\lambda}\sqrt{n_1^2 - n_2^2}$$

$$V = \frac{\pi d}{\lambda}(NA)$$

Where, d is the core diameter,

n_1 and n_2 are refractive indices of core and cladding

λ is the wavelength of light propagating through the fiber.

Number of Modes

$$M_n = \frac{V^2}{2}$$

Types of optical fibers:

Depending on NA, refractive index profile, fractional index change and modes of propagation optical fibers are classified into three types. They are

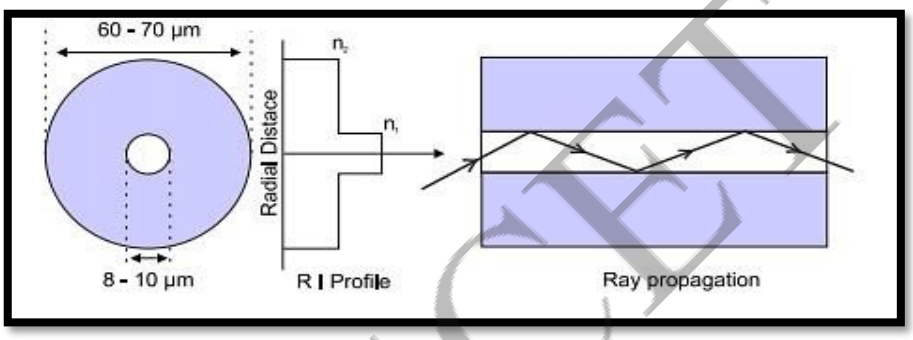
1. Step index single mode fiber
2. Step index multimode fiber
3. Graded index multimode fiber

1. Step index single-mode fiber (SISM)

❖ In steps index single mode optical fiber, the refractive index of the core is uniform. Similarly, the refractive index of the cladding is also uniform; but it is lesser than that of the cladding.

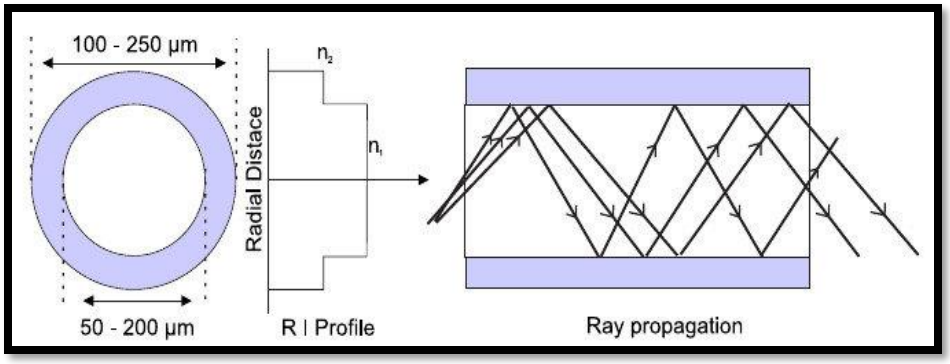
Hence R.I profile of SISM optical fiber takes the shape of a step as shown in the figure.

- The diameter of these types of fibers is about 8-10 μm and the outer diameter value of cladding is about 60-70 μm.
- Since the core diameter is very small, it can guide a single mode as shown in the diagram.Hence it is called single-mode fiber.
- The waveguide acceptance angle, NA, and fractional index change of these fibers are very small.
- The laser can be used as the source of light for operating systems involving these fibers.
- They are used in long-distance communication like in operating submarine cable systems.



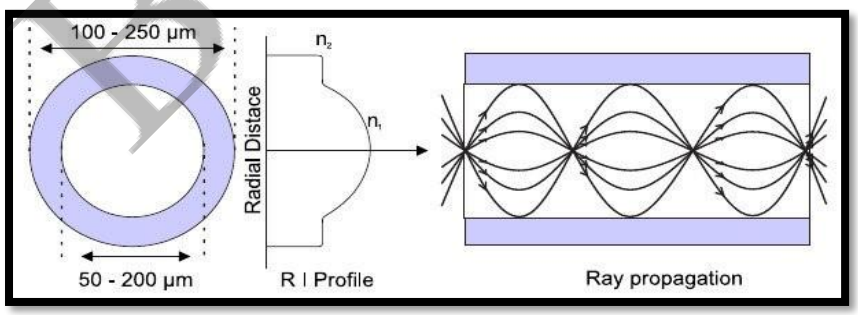
2. Step index multimode fiber (SIM):

- Its construction is similar to that of single-mode fiber but its core has a larger diameter. The core andthe cladding diameter of step-index multimode fiber is 50 to 200 μm and 100 to 250 μm respectively.
- The geometry of a step-index multimode fiber is shown in Figure.
- Because of its larger diameter, it can propagate a large number of modes as shown in the figure.
- The NA, fractional index change, and acceptance angle are very large compared to SISM.
- In SIMM fibers Laser or LED is used as a source of light.
- Its refractive index profile is also similar to that of a single-mode fiber but with a larger plane regionfor the core.
- It is the least expensive and finds its application in data link between central offices which has lowerbandwidth.
- Its core has a much larger diameter by the virtue of which it will be able to support the propagation of alarger number of modes as shown.



3. Graded index multimode fiber (GRIN):

- Its geometry is similar to that of multimode fiber i.e diameter of the Core and Cladding diameter are 50 to 200 µm and 100 to 250 µm respectively. Its geometry is as shown in the figure.
- In GRIM optical fibers, the RI of a core is not uniform. It decreases uniformly from the axis of the fiber towards the core-cladding and becomes equal to that of the cladding at the interface. But the R.I of cladding is uniform but lesser than that of the core.
- Either laser or LED is used as a source for operating the system.
- It is the most expensive; splicing can be done with some difficulty. It is used in the telephone trunkbetween the central offices.
- This fiber can propagate in several modes and propagation is more orderly as shown in the figure.



Attenuation

The loss of signal suffered by an optical signal when it propagates through the fiber is called attenuation or fiber loss. As a result output power through an optical fiber is always less than that of the input signal. It can be shown that the attenuation coefficient is given by

$$\alpha = -\frac{10}{L}\, log_{10}\left(\frac{P_{out}}{P_{in}}\right)\, dB/km$$

Where, α is the co-efficient of attenuation
L is the length of the optical fiber
P_{out} is the optical power output
P_{in} is the optical power input
-ve sign indicates the decrement in the power

Types of attenuation losses in optical fibers

Attenuation is caused by three mechanisms; they are Absorption, Scattering, and Radiation losses.

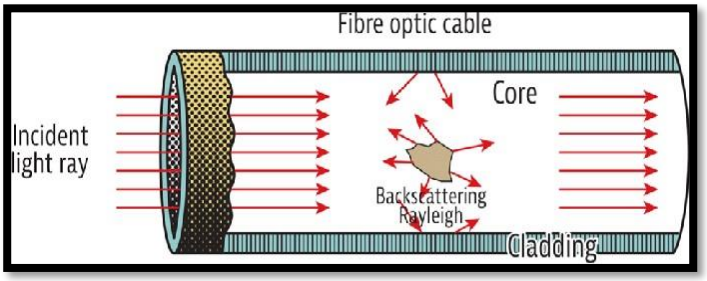
Absorption: This arises due to the absorption of photons associated with the optical signal. Theseabsorption losses are of two types. They are intrinsic losses **or** extrinsic losses.

***Intrinsic absorption*: -** The absorption of signal photons by the basic core material itself is called intrinsic loss. The molecules in the fiber absorb energy at a certain wavelength. Even a highly pure glass (core material) absorbs light energy (photons) in specific wavelength regions.

***Extrinsic absorption*:-** This is due to the absorption of light energy (photons) by the impurities present in the fiber. Electrons on absorbing energy get excited to the higher energy states. Later these electrons are de-excited into the lower energy levels by liberating energy in the form of heat. The type of impurity atoms generally present in the fibers is such as transition metal ions such as Iron, chromium, vanadium, manganese, etc. The other impurity atom which would cause significant loss is the Hydroxyl ion (OH^-) which enters into the fiber at the time of fabrication. This indicates high purity silica glass is required for communication.

These losses can be minimized by restricting metal ions content to less than a few parts/billion and OH impurity to less than a few parts per million.

Scattering: There is structural inhomogeneity and imperfection present in the fiber which enters intothe fiber during fabrication of the fiber.

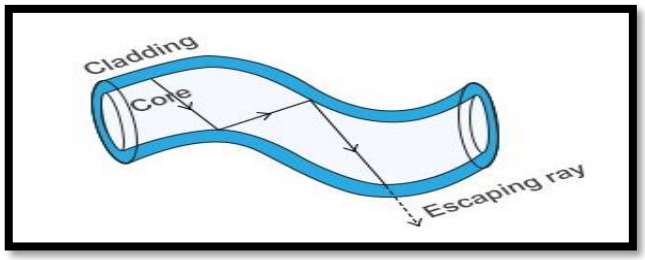


This leads to sharp variations in the refractive index of the material. Which are caused by materials whichare smaller than the wavelength of the signal. When a signal travels through the fiber, the photons may get scattered by these smaller particles. This type of scattering is the same as **"Rayleigh scattering"** (Rayleigh scattering is inversely proportional to the fourth power of wavelength [$1/\lambda^{-4}$] which means that signalsof smaller wavelengths are scattered to a larger extent. Due to Rayleigh scattering, the photons move in a random direction and leave the fiber, thus leading to loss. Thus, the loss can be minimized by using signals having larger wavelengths. It is found that scattering loss can be minimized by using a signal of wavelength more than **0.8 μm**, below which the scattering loss is very high.

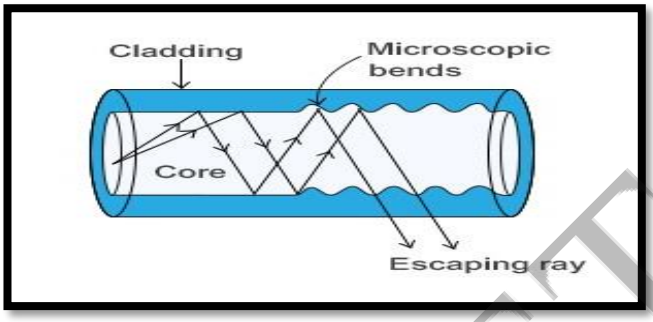
Radiation losses (Bending losses):

The radiation losses occur due to the bending of optical fibers. These bending's generally occur while turning the fiber around the corner or while wrapping the fiber. Two types of bending's are observed in optical fibers. They are macro bending and micro bending of the fiber.

Macro bending: The bending of the fiber having radii, more than the radius of optical fiber is called macro bending. If the bending is more than the critical value, may result in the modification of the angle of incidence on the core- cladding interface and hence signal fails to undergo TIR; hence radiation loss takes place as shown in Figure.



Micro bending: The bending's that occur due to non-uniformities in the manufacture of the fiber or by the lateral pressure or crushing force acting on the fiber during cabling. At these bends light rays undergo reflections. As a result, light leak out of the fiber. These losses can be minimized by covering the fiber with a compressible jacket (polyurethane jacket) over the fiber, which can withstand the stresses, and also by keeping the fiber straight.



Applications of optical fibers:

The optical fibers are widely used in various fields like communications, medical, domestic etc. Applications like point-to-point communication and fiber optic networking are discussed in detail.

Point to point Communication

- In optical fiber communication the signal can be transmitted through the fiber only in the form of optical signal.
- At first the audio signal is converted into electrical analog signals with the help of transmitter in the telephone.
- These analog signals are fed in to a coder where they get converted into binary form. This binary data is made to enter in to an optical transmitter where these get converted in to optical signals by modulating the light signals.
- These optical signals are fed in to the optical fiber within the angle of acceptance. At the output end the optical signals are fed in to a photo detector which converts them in to binary form and then in to a decoder where the signal converted into analog form and finally gets converted in to voice at the receiver end.
- As the signal propagates through the optical fiber there may be loss of signal due to various factors like as discussed in attenuation and also the signal may be distorted due to the spreading of pulses with time which is mainly because of the variation in the velocity of the different spectral components through the optical fiber.
- With distance the loss of the signal becomes more. Hence before the entire gets lost it is necessary to have repeaters which amplifies the signal but there is no such device which can directly amplify the optical signal and therefore at each repeater the optical signal needs to be converted in to electrical signal then amplify and again convert it back in to optical signal and then fed in to the fiber. This process has to be followed at every repeater, which restricts the

speed of the signal transmission.

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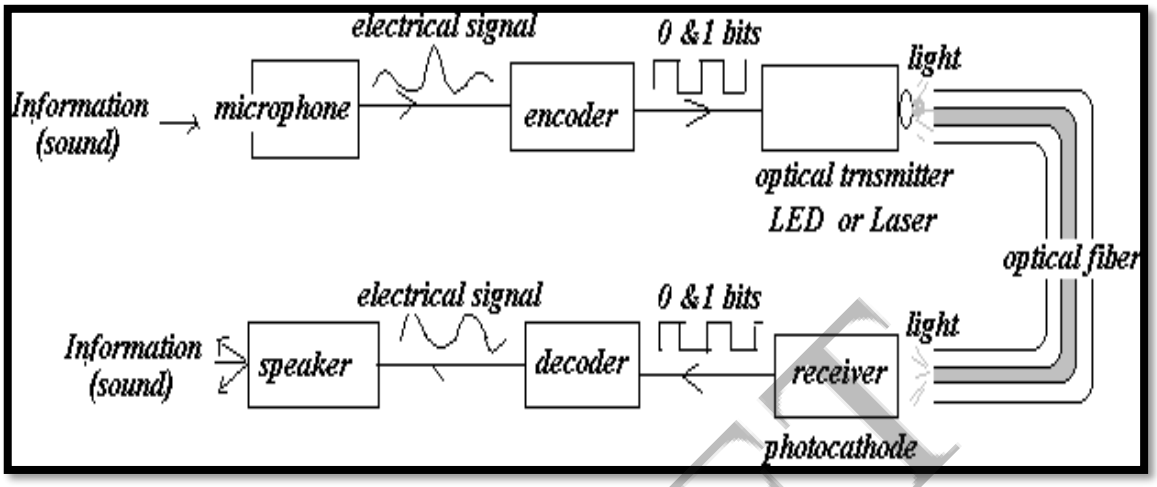


Fiber optic networking

Fiber optical networks use signals encoded onto light to transmit information among various nodes of a telecommunication network. They operate from the limited range of a local-area network (LAN) or over a wide-area network (WAN). These high-speed communication links are made up of cables containing strands of glass thinner than a human hair that transmits information such as telephone, television, and internet signals in the form of infrared light pulses. It sends rapid data transfer capability, they can also be used for artificial intelligence (AI) systems for cars or drones, to manage safety on construction sites and work areas, and to ensure fast delivery of fundamental health and public services.

A fiber optic network is made up of cables containing bundles of glass or plastic strands called optical fibers, which carry data that has been transformed into light. The light is transmitted along the fiber optic network by a laser, after having been converted by a computer into digital data signals. Light travels along the cable without “leaking out” by bouncing off the mirror-like walls of the glass or plastic. The structure of the cable also contributes to keeping the light from dispersing. The light beam travels down the core located in the middle of the cable. This is wrapped with another glass layer called “cladding.”

There are several kinds of fiber optic networks:

Fiber to the home (FTTH): the fiber optic connection starts at the telecom operator’s central office and goes all the way to the user’s home. This technology offers the best performance.

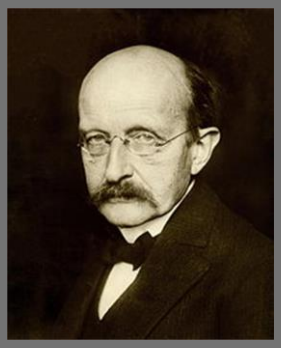
Fiber to the cabinet (FTTC): the fiber optic connection covers the section from the central office to the street cabinet, while the final connection from the cabinet to the home uses copper cables.
Fiber to the tower (FTTT): the fiber optic connection links the primary telecommunications network to the cellular network towers.

2022

Quantum Mechanics

The behavior of entities at atomic scale more pronounced

The gestation of Quantum Physics has been very long and its phenomenological foundations were various. Historically the original idea came from the analysis of the black body spectrum. This is not surprising since the blackbody, in fact an oven in thermal equilibrium with the electromagnetic radiation, is a simple and fundamental system once the laws of electrodynamics are established. As a matter of fact, many properties of the spectrum can be deduced starting from the general laws of electrodynamics and thermodynamics; the crisis came from the violation of energy equipartition. This suggested to Planck the idea of quantum, from which everything originated.



Max Planck

Department of Physics

BGS College of Engineering and Technology

VTU Syllabus: **De Broglie Hypothesis and Matter Waves, de Broglie wavelength and derivation of expression by analogy, Phase Velocity and Group Velocity, Heisenberg's Uncertainty Principle and its application (Nonexistence of electron inside the nucleus-Non Relativistic), Principle of Complementarity, Wave Function, Time independent Schrodinger wave equation, Physical Significance of a wave function and Born Interpretation, Expectation value, Eigen functions and Eigen Values, Particle inside one-dimensional infinite potential well, Waveforms and Probabilities.** Numerical problems.

Introduction to Quantum mechanics: The discoveries and insights over the last three centuries share a characteristic feature: seemingly unconnected phenomena turned out to be manifestations of the same fundamental principle. It was a period of unification of disparate fields of experience. Here are some of the most important steps. Newton showed that the motion of the planets is governed by the same law as the free fall of an object on earth. Thus, the classical theory unifies terrestrial and celestial mechanics. In contrast to the belief of the ancients, the classical theory shows that the world of the earth and the macroscopic phenomenon.

For a long time, the phenomena of electricity, magnetism, and light appeared to be unconnected. In the first half of the nineteenth century, one of the great unifications of physics took place. Faraday and Maxwell, together with many others, were able to show that all three phenomena are manifestations of the electromagnetic field. And so, the field concept entered into physics. The simplest example is the electric field of an electric charge that exerts a force on another charge when the latter falls within its range. An electric current produces a magnetic field that exerts a force on magnetic materials. In Quantum mechanics the fundamental concepts were not too different from those of our everyday experience, such as particle, position, speed, mass, force, energy, and even field. We often refer to those concepts as classical. The world of atoms cannot be described and understood with those concepts. For atoms and molecules, the ideas and concepts formed in dealing with the objects in our immediate environment no longer suffice. One needs a new concept to understand the properties of atoms. The quantum mechanics changed our old concepts of reality in many respects.

According to Max Planck the energy could be taken only certain discrete values as follows

i. A System can absorb and emit the radiations in discrete packets called quanta.
ii. If 'ν' is the frequency of an electromagnetic oscillator. Then its energy is proportional to ν.

i.e., $E = h\nu, 2h\nu, 3h\nu, \ldots\ldots$

Where h is Planck's constant and its value is 6.625×10^{-34} JS.

In other words, it states that exchange of energy between the radiation and matter cannot takes place continuously. The energies of the atoms are said to be **quantized** and the allowed energy levels are called as **quantum levels**.

$$E = nh\nu$$

A Black body which absorbs all radiations and emits all radiations irrespective of wavelengths can be explained only based on quantum mechanical principles. Then, the energy density corresponds to the emitted radiation in the wavelength range of λ and $\lambda + d\lambda$ is given by

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{\left[e^{\frac{hc}{\lambda kT}} - 1\right]}$$

Wave nature of particles: The light, believed to be a wave, exhibited particle properties and that electrons, believed to be particles, exhibited wave properties. Let us call it the wave-particle duality. Louis de-Broglie extended the wave particle duality of light to all the fundamental entities of physics such as electrons, protons neutrons etc. de-Broglie put a bold suggestion that, if radiation can behave as particle under certain circumstances, then, one can even expect that entities which ordinarily behaves as particles to exhibit wave properties under appropriate circumstances.

The hypothesis of de-Broglie was that the dual nature that is wave –particle behaviour of radiation applies equally well to matter. Just as a photon has a light wave associated with it, governs its motion. Such a wave associated with matter are called matter waves or de-Broglie waves and the hypothesis is called de-Broglie hypothesis.de-Broglie hypothesis opened up a new thinking in almost all the fields of Physics. In fact, it can be treated as the new beginning of the Modern Physics.

The expression of the wavelength associated with a material particle can be derived on the analogy of radiation as follows:

Considering the Plank's theory of radiation, the energy of a photon (quantum) is given by

$$\mathbf{E = h\nu} \qquad \textbf{and} \qquad \mathbf{E = mc^2}$$

Equating above equations, we get

$$\boldsymbol{h\nu} = \mathbf{mc^2}$$

$$\frac{\boldsymbol{hc}}{\boldsymbol{\lambda}} = \mathbf{mc^2}$$

$$\boldsymbol{\lambda} = \frac{\mathbf{h}}{\mathbf{mc}}$$

Where cis the speed of light and h= 6.62×10^{-34} JS is Planck's constant.

The momentum of a particle of mass 'm' and velocity 'v' is $\mathbf{p} = \mathbf{mv}$ and its de-Broglie wavelength is accordingly

$$\boldsymbol{\lambda} = \frac{\mathbf{h}}{\mathbf{mv}}$$

This physically means that a material particle of mass 'm' moving with the velocity 'v' has a wave associated with it of de-Broglie wavelength.

If 'E' is the kinetic energy of the particle, then $\because$ De-Broglie wavelength is $\boldsymbol{\lambda} = \frac{\boldsymbol{h}}{\sqrt{\boldsymbol{2mE}}}$

For an accelerated charged particle at a potential difference of 'V', the de-Broglie's wavelength is

$$\boldsymbol{\lambda} = \frac{\boldsymbol{h}}{\sqrt{\boldsymbol{2meV}}}$$

Heisenberg's Uncertainty Principle: To regard a moving particle as a wave group implies that there are fundamental limits to the accuracy with which we can measure such "particle" properties as position and momentum. According to Heisenberg's uncertainty principle: It is impossible to know both the exact position and momentum of a particle at the same time, It states that "The product of inherent uncertainties involved in the measurements position and momentum simultaneously is greater than or equal to $\boldsymbol{h}/\boldsymbol{4\pi}$".

$$\Delta x.\Delta p \geq \frac{h}{4\pi}$$

Another form of the uncertainty principle concerns energy and time is given by

$$\Delta E.\Delta t \geq \frac{h}{4\pi}$$

The Uncertainty principle between angular displacement and angular momentum is given by

$$\Delta J.\Delta \emptyset \geq \frac{h}{4\pi}$$

Significance of Uncertainty principle: The Physical Significance of this principle is that one should not think of exact position or an accurate value of momentum of the particle. Instead, one should think of probability of finding the particle in a certain region or the probability of finding the momentum of the particle. The estimation of such probabilities is made by means of certain mathematical function called wave function in quantum mechanics.

Applications of uncertainty principle:

Non-existence of electrons in atomic nucleus:

We know that, the diameter of nucleus cannot exceed $1 \times 10^{-14}\ m$. Let us assume that, if an electron exists inside the nucleus. This means, Uncertainty in position cannot exceed the diameter of the nucleus. Hence $\Delta x \leq 1 \times 10^{-14} m$.

We know from Heisenberg uncertainty principle

$$\Delta x.\Delta p \geq \frac{h}{4\pi}$$

$$\Delta p \geq \frac{6.626 \times 10^{-34}}{4 \times 3.14 \times 1 \times 10^{-14}}$$

$$\Delta p \geq 5.2754 \times 10^{-21}\ \text{kgms}^{-1}$$

Therefore, the uncertainty in momentum of the electron is $5.2754 \times 10^{-21}\ \text{kgms}^{-1}$ and hence, the momentum of an electron is equal to uncertainty in its momentum.

$$p \geq 5.2754 \times 10^{-21}\ \text{kgms}^{-1}$$

Then the relativistic energy of the electron in the nucleus is given by

$$E = \frac{p^2}{2m}$$

Where m is the rest mass of the electron and p is the momentum

$$E = \frac{[5.2754 \times 10^{-21}]^2}{[2 \times 9.1 \times 10^{-31}]}$$

$$E = 1.5291 \times 10^{-11}\text{J} \qquad \text{or}$$

$$E = 95.572\ \text{MeV}$$

The value indicates that an electron requires energy of 95.572 MeV to exist inside the nucleus. But the measurements made in the beta decay shows that the kinetic energy of the order of 3 – 4 MeV. The actual energy of the electrons exists in various orbits outside the nucleus is less than the value 95.572 MeV, if it exists inside the nucleus. Therefore, it is clear the no electron can exist inside the nucleus.

Principle of Complementarity:

Statement: In 1928, Bohr stated as "In a situation where the wave aspect of a system is revealed, its particle aspect is concealed; &, in a situation where the particle aspect is revealed, its wave aspect is concealed. Revealing both simultaneously is impossible; the wave & particle aspects are complementary".

Explanation: We know that the consequence of the uncertainty principle is both the wave and particle nature of the matter cannot be measured simultaneously. In other words, we cannot precisely describe the dual nature of light.

- If an experiment is designed to measure the particle nature of the matter, during this experiment, errors of measurement of both position & the time coordinates must be zero and hence the momentum, energy & the wave nature of the matter are completely unknown.
- Similarly, if an experiment is designed for measuring the wave nature of the particle, then the errors in the measurement of the energy & the momentum will be zero, whereas the position and the time coordinates of the matter will be completely unknown. From the above explanation, we can conclude that, when the particle nature of the matter is measured or displayed, the wave nature of the matter is necessarily suppressed and vice versa.

Phase Velocity:

The velocity with which individual wave travels in group of waves is called Phase velocity.

$$v_{ph} = \frac{\omega}{k}$$

Group Velocity:

The velocity with which resultant envelop of the group of waves travel is called group velocity.

$$v_g = \frac{d\omega}{dk}$$

Wave function:

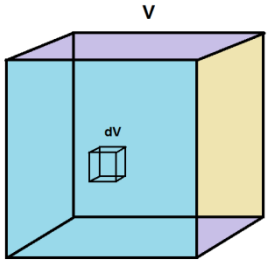
We know that matter exhibits wave like behavior under certain conditions. When the momentum of the particle is well defined, the wave can be of infinite extent. Therefore, a particle moving along x-axis with well-defined momentum is described by an infinite plane wave $\psi(x, t)$and is given by

$$\psi(x, t) = A\, e^{[i(kx-\omega t)]}$$

It is a function of space variables (x, y, z) and time 't'

Where ω angular frequency and k is wave vector

In the case of electromagnetic waves, the electromagnetic waves vary with space and time. In case of sound waves, it is describing pressure variation in space and time. In other words, to describe the wave function one requires quantity which varies in space and time. In analogy with these, to describe the matter waves associated with the particle motion, one requires a quantity which varies with space and time, this variable quantity is called as wave function ψ(x, t). The solution describes wave aspect, particle aspect in consistent with the uncertainty principle.



Physical significance of wave function:

The wave function ψ(x, t) signifies the probability of finding the particle described by the wave function at the point $\vec{r} = (x, y, z)$ and at time 't'

Properties of wave function:

The knowledge of the wave function of the particles can be evaluated by knowing the values of $\psi(x, t)$ from the Schrödinger wave equation.

i) **The wave function $\psi(x, t)$ is single-valued everywhere:** The wave function should have single value rather than multiple values. Consider a wave function $\psi(x, t)$ has an interval of wave function ψ_1, ψ_2 and ψ_3.

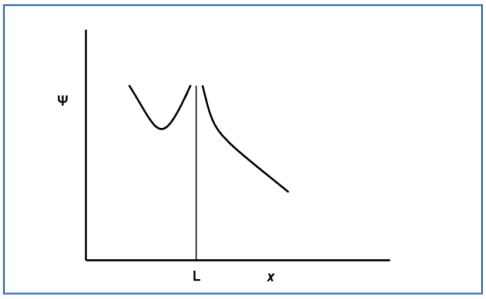
ii) **Wave function is finite everywhere:** Consider a wave function ψ as a function of x as shown in the figure 1.4. The wave function is infinite at $x = L$, Which means the probability of finding the particle is infinitely large at $x = L$. This violates the Heisenberg uncertainty principle. Thus, the wave function becomes unacceptable. Therefore, the wave function must have a finite value everywhere.

iii) **Wave function and its derivatives are continues everywhere:** When the wave function is continuous and finite everywhere, the first and second derivatives are also continuous which is essential to evaluate the Schrödinger equation. Hence wave functions must be continuous everywhere.

iv) It is large in magnitude where the particle is likely to be located and small in other places.

v) The wave function describes the behavior of a single particle not the statistical distribution of a number of such quanta.

vi) It can interfere with itself (phenomenon of electron diffraction)



Probabilitydensity/BornInterpretation:

Consider a particle whose position is independent of time in a volume V inside which particle is known to be present in a volume element dV. Let ψ be the wave function associated with the particle, and then the square of the wave function associated with the particle is interpreted as measure of probability density.

That is $|\psi|^2$ is the probability density per volume elementdV that the particle will be found in that volume element.

$$P(x, t) = |\psi|^2 \, dV$$

If the wave function is complex, then, the probability density is product of wave functions ψ and its complex conjugate ψ^*, therefore the probability density is given by

$$P(x,t) = \psi\,\psi^*$$

Normalization: If the particle exists somewhere at all the time. Let ψ be the wave function associated with the particle, then the probability density of particle in a volume element is $|\psi|^2\ dV$. If we further extend the case where the particle is definitely present in some region or space, as per the statistical rule the normalization is given by.

$$\int_0^v |\psi|^2\ dV = 1$$

This is based on the fact that, a value 1 for probability means, it is clearly a certainty. However, in the above case if we are not at all certain about locating the particle in a finite volume anywhere in the space, then the expectation will become limited to the extent that it exists somewhere in space. Then the limits in equation extend to $-\infty$ to $+\infty$ and the probability becomes.

$$\int_{-\infty}^{+\infty} |\psi|^2\ dV = 1$$

This process is called Normalization.

Expectation Value:

In quantum mechanics "The expectation value is the probabilistic expected value of the result(measurement) of an experiment. It can be thought of as an average of all the possible outcomes of a measurement as weighted by their likelihood." Expectation value as such it is not the most probable value of a measurement. In the real sense the expectation value may have Zero probability of occurring. Let us consider a particle moving along the X-axis. The result of a measurement of the position x is a continuous random variable. Consider a wave function ψ(x,t). The $|\ \psi(x,t)\ |^2$ value is a probability density for the position observable and $|\ \psi(x,t)\ |^2$ dx is the probability of finding the particle between x and x+ dx at time t. Thus, if a measurement of position is repeated many times in identical circumstances, many possible outcomes are possible and the expectation value of these outcomes is, according to the following equation $< x >= \int_{-\infty}^{\infty} x\,|\ \psi(x,t)\ |^2\ dx$.

Schrodinger's time independent wave equation:

The wave function of matter wave which is confined to a small region of space is given by

$$\psi(x,t) = A\ e^{[i(kx-\omega t)]} \qquad (1)$$

In many cases the potential energy does not depend on time 't' explicitly. In these cases, there is no external force acts on the particle. Hence, the potential energy depends on the position of the particle; the wave function for such cases can be obtained as follows.

Differentiating the equation (1) twice with respect to '*x*' we get

$$\frac{d^2\psi}{dx^2} = -k^2\psi \qquad (2)$$

Differentiating the equation (1) twice with respect to '*t*' we get

$$\frac{d^2\psi}{dt^2} = -\omega^2\psi \qquad (3)$$

We have Phase velocity $v_{phase} = \frac{\omega}{k}$ or $k = \frac{\omega}{v_{phase}}$ (4)

Substituting equation (4) in (2), we get

$$\frac{d^2\psi}{dx^2} = -\frac{\omega^2}{v_{phase}^2}\psi \qquad (5)$$

Using (3) in (5), $\frac{d^2\psi}{dx^2} = \frac{1}{v_{phase}^2}\frac{d^2\psi}{dt^2}$

This is the equation for the travelling wave. Now, we have $k = \frac{2\pi}{\lambda}$ (6)

Substituting equation (6) in (2), we get

$$\frac{d^2\psi}{dx^2} = -\frac{4\pi^2}{\lambda^2}\psi \qquad (7)$$

The total energy of the particle is

$$E = KE + PE \quad \text{or}$$

$$E = \frac{P^2}{2m} + V_x \qquad (8)$$

Where, V_x is the potential energy, it depends only on position and is independent of time.

The De-Broglie's wave length is $\lambda = \frac{h}{P}$ or $P = \frac{h}{\lambda}$ (9)

Substituting equation (9) in (8), we get

$$E - V_x = \frac{h^2}{2m\lambda^2}$$

Or $\frac{2m\,(E-V_x)}{h^2} = \frac{1}{\lambda^2}$ (10)

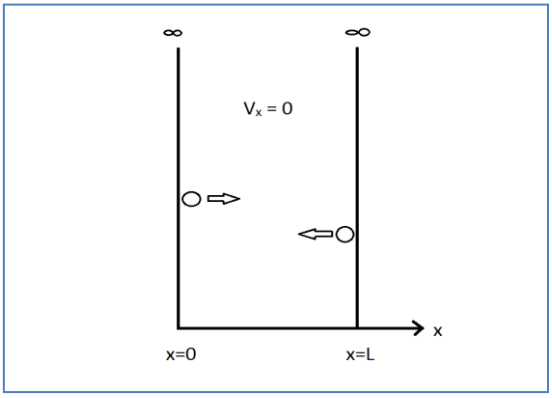
Substituting equation (10) in (7), we get

$$\frac{d^2\psi}{dx^2} = -\frac{8\pi^2 m\,(E-V_x)}{h^2}\psi$$

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m\,(E-V_x)}{h^2}\psi = 0$$

This is the Expression for Schrodinger's time-independent equation.

Particle in one dimensional potential well of infinite height



Consider particle confined in a one-dimensional potential well of infinite height and of length '*L*' as shown in the figure.1.5

a) The particle moves along x – axis between x =0 and x = L exhibits elastic collisions
b) The particle inside the potential well is not influenced by any external force. Therefore, potential energy is minimum and taken to be zero.

$$V_x = 0 \ for \ 0 < x < L$$

c) The potential energy outside the potential well is infinite.

$$V_x = \infty \ for \ 0 > x > L$$

Consider Schrödinger time independent equation

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m\,(E - V_x)}{h^2}\psi = 0 \qquad (1)$$

For the particle inside the potential well potential energy

$$\text{i.e. } V_x = 0 \ for \ 0 < x < L$$

Substituting in (1) we get

$$\frac{d^2\psi}{dx^2} + \frac{8\pi^2 m\,E}{h^2}\psi = 0 \qquad (2)$$

$$\text{Let} \quad \frac{8\pi^2 m\,E}{h^2} = k^2 \qquad (3)$$

Then equation (2) becomes

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$

The solution of the above equation is given by

$$\psi(x,t) = A\,coskx + B\,sinkx \qquad (4)$$

Where A and B are constants depends on boundary conditions

a) When $x = 0$; $\psi(x,t) = 0$

equation (4) becomes $\quad 0 = A\,cos\,0 + B\,sin\,0$

$$A = 0$$

Substituting $A = 0$ in equation (4)

$$\psi(x,t) = B\ sinkx \qquad (5)$$

b) When $x = L$; $\psi(x,t) = 0$

Equation (5) becomes $0 = B\ sinkL$

On solving $B \neq 0$ or $sinkL = 0$

$$kL = n\pi$$

$$\therefore \quad k = \frac{n\pi}{L} \qquad (6)$$

Substituting equation (6) in (3), we get

$$\frac{n^2\pi^2}{L^2} = \frac{8\pi^2 m\, E}{h^2}$$

$$E_n = \frac{n^2h^2}{8mL^2} \qquad (7)$$

The above equation is called the Eigen energy values for a particle in one dimensional potential well.

For $n = 1$ $$E_1 = \frac{h^2}{8mL^2}$$

This is the least possible energy possessed by the particle in a 1-D potential well called Zero-point energy.

For $n = 2$ $$E_2 = 4E_1$$

This is the energy of the particle in first excited state and so on.

Normalization: For the particle inside the potential well, the probability of finding a particle is unity and is given by.

$$\int_0^L |\psi|^2\ dx = 1$$

$$\int_0^L B^2 sin^2 kx\, dx = 1$$

$$\int_0^L B^2 \left[\frac{(1 - cos2kx)}{2}\right] dx = 1$$

$$\int_0^L \frac{B^2}{2} dx - \int_0^L \frac{B^2}{2}\ cos2kx\, dx = 1$$

On simplification we get

$$B = \sqrt{\frac{2}{L}}$$

Substituting $B = \sqrt{\frac{2}{L}}$ and $k = \frac{n\pi}{L}$ in (5)

$$\psi_n(x,t) = \sqrt{\frac{2}{L}} \quad sin\frac{n\pi x}{L}$$

This equation is called as Eigen wave function, where $n = 1, 2, 3, 4 \ldots \ldots$

Eigen energy values and Eigen function for a particle in infinite potential well:

We have equation for the Energy Eigen values

$$E_n = \frac{n^2h^2}{8mL^2}$$

and the Eigen wave function is

$$\psi_n(x,t) = \sqrt{\frac{2}{L}} \quad sin\frac{n\pi x}{L} \qquad (8)$$

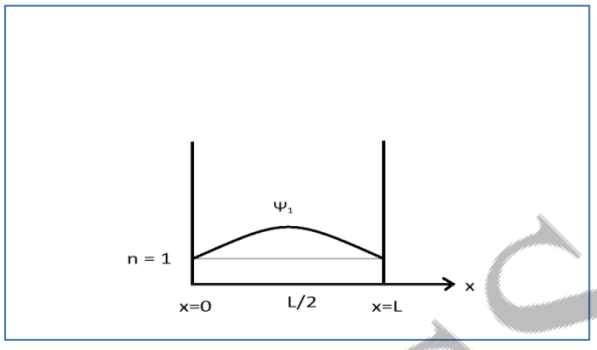


Figure 1.6

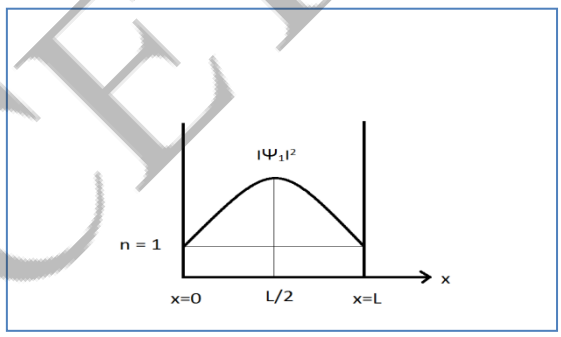


Figure 1.7

Case i When $n = 1$ for ground stand, the wave function and corresponding energy for a particle in ground state is

$$E_1 = \frac{h^2}{8mL^2}$$

$$\psi_1(x,t) = \sqrt{\frac{2}{L}} \quad sin\frac{\pi x}{L}$$

E_1 is called the ground state energy or Zero-point energy and ψ_1 is the wave function corresponds to particle in ground state. At the boundary conditions $x = 0$ and $x = L$, the wave function $\psi_1 = 0$. It indicates the probability of finding the particle at both $x = 0$ and

$x = L$ is zero i.e., $|\psi_1|^2 = 0$. However, the probability of finding the particle is maximum at the centre i.e., $x = \frac{L}{2}$. The physical representation of the normalized wave function and probability density of wave function of particle at $n = 1$ is shown in below figures 1.6. and figure 1.7.

Case ii When $n = 2$, the particle in the excited state known as first excited state. Therefore, the energy and the wave function corresponding of the particle in the first excited state is given by

$$E_2 = \frac{4h^2}{8mL^2}$$

$$\psi_2(x,t) = \sqrt{\frac{2}{L}} \quad sin\frac{2\pi x}{L}$$

The normalized wave function and probability density of wave function of the particle in the first excited state is as shown in the figure 1.8. and 1.9.

The probability of finding the particle in the first excited state at $x = 0$, $x = \frac{L}{2}$ and $x = L$ is zero i.e., $|\psi_2|^2 = 0$. Since $\psi_2 = 0$. However, the probability of finding the particle is maximum at $x = \frac{L}{4}$, and $x = \frac{3L}{4}$.

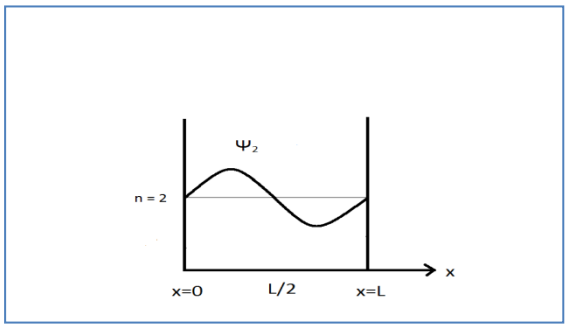


Figure 1.8

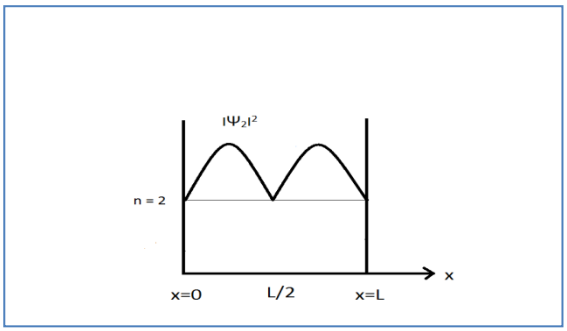


Figure 1.9

Reference:

1. Concepts of Modern Physics: Arthur Beiser. McGrawhill company. 2009.
2. Modern Physics: Robert Resnic, Robert Eiseberg. John-Wiley.2000.
3. Engineering Physics by Rajendran, McGrawhill company. 2011.

David Elieser Deutsch (born on 18 May 1953)is a British physicist at the University of Oxford. He is a Visiting Professor in the Department of Atomic and Laser Physics at the Centre for Quantum Computation (CQC) in the Clarendon Laboratory of the University of Oxford. He pioneered the field of quantum computation by formulating a description for a quantum Turing machine, as well as specifying an algorithm designed to run on a quantum computer. He has also proposed the use of entangled states and Bell's theorem for quantum key distribution and is a proponent of the many-worlds interpretation of quantum mechanics. His work on quantum algorithms began with a 1985 paper, later expanded in 1992 along with Richard Jozsa to produce the Deutsch–Jozsa algorithm, one of the first examples of a quantum algorithm that is exponentially faster than any possible deterministic classical algorithm. *The Fabric of Reality* was shortlisted for the Rhone-Poulenc science book award in 1998.Deutsch was awarded the Dirac Prize of the Institute of Physics in 1998, and the Edge of Computation Science Prize in 2005.In 2017, he received the Dirac Medal of the International Centre for Theoretical Physics (ICTP). Deutsch is linked to Paul Dirac through his doctoral advisor Dennis Sciama, whose doctoral advisor was Dirac. Deutsch was elected a Fellow of the Royal Society (FRS) in 2008.In 2020 he was awarded an Honorary Fellowship of the Cybernetics Society. In 2018, he received the Micius Quantum Prize. In 2021, he was

Module-3

Quantum Computing:

Principles of Quantum Information & Quantum Computing:
Introduction to Quantum Computing, Moore's law & its end, Differences between Classical & Quantum computing. Concept of qubit and its properties. Representation of qubit by Bloch sphere. Single and Two qubits. Extension to N qubits.

Dirac representation and matrix operations:

Matrix representation of 0 and 1 States, Identity Operator I, Applying I to|0⟩and |1⟩ states, Pauli Matrices and its operations on |0⟩and |1⟩states, Explanation of i) Conjugate of a matrix and ii) Transpose of a matrix. Unitary matrix U, Examples: Row and Column Matrices and their multiplication (Inner Product), Probability, and Quantum Superposition, normalization rule. Orthogonality, Orthonormality. Numerical Problems

Quantum Gates:

Single Qubit Gates: Quantum Not Gate, Pauli – X, Y and Z Gates, Hadamard Gate, Phase Gate (or S Gate), T Gate
Multiple Qubit Gates: Controlled gate, CNOT Gate, (Discussion for 4 different input states). Representation of Swap gate, Controlled -Z gate, Toffoli gate.

MODULE-3

QUANTUM COMPUTING

3.1. Principles of Quantum Information & Quantum Computing

3.1.1. Introduction to Quantum Computing

A remarkable integration of quantum physics, computer science, information theory and is known as quantum computing. The goal of quantum computing is to alter the notion of computation rather changing the physical substrate on which computation is carried out from classical to quantum. The fundamental unit of computation is no longer the bit, but rather the **quantum bit or qubit**. By basing computation on a quantum mechanical framework, new cryptographic techniques, faster algorithms, and effective communication protocols have been established.

- Quantum computing is a modern way of computing that is based on the science of quantum mechanics and its unbelievable phenomena. It is a beautiful combination of physics, mathematics, computer science and information theory.
- It provides high computational power, less energy consumption and exponential speed over classical computers by controlling the behavior of small physical objects i.e., microscopic particles like atoms, electrons, photons, etc.
- Here, we present an introduction to the fundamental concepts and some ideas of quantum computing. The basic working of quantum computing and the quantum properties it follows like superposition, entanglement and interference.
- Before focusing on the significances of a general-purpose quantum computer and exploring the power of the new arising technology, it is better to review the origin, potentials, and limitations of the existing traditional computing. This information helps us in understanding the possible challenges in developing exotic and competitive technology. It will also give us an insight into the ongoing progress in this field.

3.1.2. Moore's law& its end:

Gordon Moore, Co-founder of Intel indicated that "the number of transistors on a microchip will double about every two years, while the cost of computers is cut into half". This suggests the exponential growth of chip capabilities and immense downward pressure on chips.

Statement: "Moore's law states that the number of transistors on a microchip will double about every two years, while the cost of computers is cut into half".

End of Moore's law: Now a days nanometer circuits are attained further reduction in the size of transistors is extremely challenging. Computer systems may have reached their limit in transistor capacity and power. Hence, industry leaders are asserting that Moore's law has come to an end, and computers will no longer have many more transistor every year.

James R. Powell calculated that due to the uncertainty principle alone Moore's law will be end by 2036.

3.1.3. Difference between Classical and quantum computing:

Sl.No.	Classical computing	Quantum computing
1.	Classical computation uses bits '0' and '1' to measure and extract information	Quantum computation use qubits to measure and extract information.
2.	Classical computers which can store either '0' or '1'	Qubit store multiple values at the same time
3.	Classical algorithms computation allows N operations.	Quantum algorithms computation allows $\sqrt{N}$operations.
4.	Classical computation employs stream of binary electrical pulses of '0' and '1'	A quantum computation uses subatomic particles such as electrons, Photons (quantum Particles for representing qubits.
5.	Classical computation allows the binary electrical pulses only in one state '0' and '1'. No mixed state	Quantum computation allows the quantum particles exists move than one state at the same time.

3.1.4. Concept of qubit and its properties

The bit is the fundamental concept of classical computation and information. Quantum computation and information use an analogous concept, ***quantum bit***, or ***qubit***. Just like classical bit has a state either 0 or 1, a qubit also has a state. The two possible states for a qubit are $|\mathbf{0}\rangle$ and $|\mathbf{1}\rangle$ which correspond to the states **0** and **1** for a classical bit. These two states are known as computational basis states.

Ex. The two states of an electron orbiting a single atom.

A qubit can be in a state other than |0⟩ and |1⟩

The difference between qubits and classical bits is that a qubit can be in a linear combination or superposition of the two states |0⟩ and |1⟩ as:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Where α and β are coefficients of both the states

Properties of Qubits:

Mathematical representation of qubit:

A Qubit mathematically represented by two states |0⟩ and |1⟩.

$$\text{i.e., } |\mathbf{0}\rangle = \begin{pmatrix}1\\0\end{pmatrix} \text{ and } |\mathbf{1}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

|0⟩ and |1⟩ are the column matrices.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

The most important properties of qubits are:

1) Superposition – The ability of existence of two states 0 and 1 at the same time in a one state. Superposition means showing all the possible states of the qubits.

2) Entanglement – The quantum measurements on qubits are perfectly correlated and they cannot be untangled into two individual qubits**.**

3) Interference – Quantum interference is a byproduct of superposition, is that it allows to bias the measurement of qubit towards a discrete state or set of states.

4) Tunneling - Which means that phenomenon of quantum particle ca travel to the opposite side of a barrier which is impossible by classical particles. Quantum tunneling allows quantum computers the ability to perform tasks faster than classical computers.

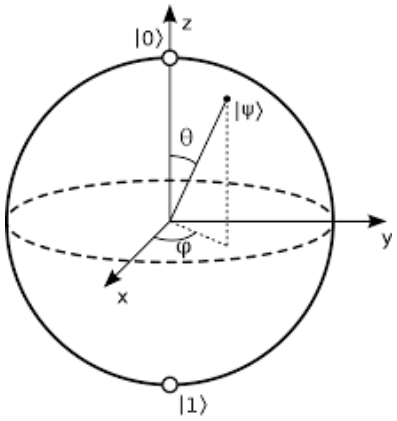
3.1.5. Representation of qubit by Bloch sphere

In quantum computing, a state of a qubit is represented on a sphere is called Bloch sphere. It is a sphere of unit radius.

Only single qubit quantum states can be represented by Bloch sphere. The state of a qubit is represented by a vector

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\Phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

Where, θ and Φ define a point on the Unit three-dimensional sphere as shown in figure of Bloch sphere.



State of Qubit on the Bloch sphere

For **θ =0** and **Φ = 0**, The state |ψ⟩ corresponds to |0⟩ and it is along **+ Z- axis.**

For **θ =180** and **Φ = 0**, The state |ψ⟩ corresponds to |1⟩ and it is along **- Z- axis.**

When **θ = 90°** and **Φ = 90°**, $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$ is a superposition state along **+ Y-axis.**

When **θ = -90°** and **Φ = 90°**, $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$ is a superposition state along **- Y-axis.**

When **θ = 90°** and **Φ = 0°**, $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ is a superposition state along **+ X-axis**.

When **θ = -90°** and **Φ = 0°**, $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ is a superposition state along **- X-axis.**

3.1.6. Single qubit: It is a vector $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ premised by two complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. Operation on qubit must preserve this norm under described by 2 x 2 unitary matrix.

The basis states of single qubit are |0⟩ and |1⟩ or linear combination of both states.

$\alpha|0\rangle + \beta|1\rangle$ are mathematically represented as

$$|\mathbf{0}\rangle = \begin{pmatrix}1\\0\end{pmatrix} \text{ and } |\mathbf{1}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

Binary representation:

$$|0\rangle = 0 \text{ X } 2^0 = 0 \text{ X } 1 = 0 \qquad \{|0\rangle\ , |1\rangle\}$$

$$|1\rangle = 1 \text{ X } 2^0 = 1 \text{ X } 1 = 1 \qquad \{0\ \ , 1\}$$

Take as all basis as column of matrix then we get an identity matrix $I = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

The most general state |ψ⟩ of a single qubit is represented by a vector of the form

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\theta}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

A single qubit represented by operators on the two-dimensional vector space as 2 X 2 matrices.

Ex. $\sigma_x = \sigma_1 = X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ $\sigma_y = \sigma_2 = Y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$ $\sigma_z = \sigma_3 = Z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

All Pauli matrices are single qubit state vector operations.

Two qubits: It is a vector represented by $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$, and the state is entangled with these number of particle values. α_0, α_1, β_0, β_1.

$$|\psi\rangle = (\alpha_0|0\rangle + \alpha_1|1\rangle)(\beta_0|0\rangle + \beta_1|1\rangle)$$

The above state is called as EPR (Einstein, Podolsky and Rosen)

The basis states of two qubit quantum gates are,

$|00\rangle$, $|01\rangle$,$|10\rangle$ and $|11\rangle$

The binary notation of the basis vector is written as follow

$|00\rangle = 0 \text{ X } 2^1 + 0 \text{ X } 2^0 = 0 + 0 = 0$ $|01\rangle = 0 \text{ X } 2^1 + 1 \text{ X } 2^0 = 0 + 1 = 1$

$|10\rangle = 1 \text{ X } 2^1 + 0 \text{ X } 2^0 = 2 + 0 \text{ X } = 2$ $|11\rangle = 1 \text{ X } 2^1 + 1 \text{ X } 2 = 2 + 1 \text{ X } 1 = 3$

Take as all basis as column of matrix then we get an identity matrix

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = I_4 = D\,[1111]$$

N – qubit: N –qubit is quantum state vector represented by

$$|\psi\rangle = (\alpha_0|0\rangle + \alpha_1|1\rangle)(\beta_0|0\rangle + \beta_1|1\rangle) \ldots\ldots\ldots\ldots..(\delta_0|0\rangle + \delta_1|1\rangle)$$

in the computational basis will label the 2^n basis vectors inn the Dirac notation using binary string of length n.

$$|00\ldots\ldots00\rangle, |01\ldots\ldots01\rangle, |10\ldots\ldots10\rangle, |11\ldots\ldots\ldots11\rangle$$

$$|00\ldots\ldots00\rangle = \begin{pmatrix}1\\0\\.\\.\\.\\.\\0\end{pmatrix} \quad |01\ldots\ldots01\rangle = \begin{pmatrix}0\\1\\0\\.\\.\\.\\0\end{pmatrix} \quad \text{etc.}$$

$$I_8 = \begin{pmatrix} 1 & 0 & . & . & . & . & 0 \\ 0 & 1 & . & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & 1 \end{pmatrix}_{nXn}$$

3.2. Dirac representation and matrix operations

The state of quantum mechanical system is described or represented by a wave function ψ (x, t) or as a state vector. Wave function is a Schrodinger representation of quantum mechanical system (more generalized form) state vector is a matrix representation (Heisenberg) of quantum mechanical system.

Dirac Bra and Ket notations:

Wave function ψ (x) is represented as a column vector called as et vector $|\psi\rangle$ and read as " **ket psi**". A state vector or $|\psi\rangle$ is written in matrix form as

$$|\psi\rangle = \begin{pmatrix} \psi(x_1) \\ \psi(x_2) \\ . \\ . \\ \psi(x_n) \end{pmatrix}.$$

The notation $|\ \rangle$ indicates that the object is a vector. $|\ \rangle$ is called as "ket vector"

The complex conjugate of ket vector is represented by "**Bra vector** ". $\langle\psi|$ read as "bra psi".

Hence the bra vector is a transpose of complex conjugate of ket vector & vice versa

$$\langle\psi| = [\psi^*(x_1), \psi^*(x_2) \ldots\ldots\ldots\ldots \psi^*(x_n)]$$

Hence ket and bra vectors are transpose of complex conjugate of each other.

The scalar or inner product of bra and ket vectors can be represented by matrix as

$$\langle\psi|\ \psi\rangle = [\psi^*(x_1), \psi^*(x_2) \ldots\ldots\ldots\ldots \psi^*(x_n)] \cdot \begin{pmatrix} \psi(x_1) \\ \psi(x_2) \\ . \\ . \\ \psi(x_n) \end{pmatrix}.$$

$$\langle\psi|\ \psi\rangle = [\psi^*(x_1)\ \psi(x_1) + \psi^*(x_2)\ \psi(x_2) + \ldots\ldots\ldots\ldots \psi^*(x_n)\ \psi(x_n)]$$

Which is a scalar quantity and matrix representation of wave function in quantum mechanics.

3.2.1. Matrix representation of 0 and 1 states:

The state of a qubit in quantum computation is represented by |0⟩ and |1⟩.

The matrix form as: $|\mathbf{0}\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|\mathbf{1}\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

3.2.2. Representation of Identity Operator:

In quantum mechanics identity operator is represented by identity matrix ($\hat{I}$) with cap read as identity operator

$$\hat{I} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}_{2X2}$$

Operation of $\hat{I}$ on |0⟩ and |1⟩ states:

Consider |0⟩ and |1⟩ states expressed in matrix form as

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ and identity operator $\hat{I} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

Let $\hat{I}|0\rangle = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1+0\\0+0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = |0\rangle$ …… (1)

$\hat{I}|1\rangle = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0+0\\0+1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = |1\rangle$ ……(2)

From the above proof of equation (1) and (2) it can be concluded that when Identity operator $\hat{I}$ operates on any qubit, basis states left unchanged. The $\hat{I}$ operators doesn't change the state of the systems.

Hence when identity operator operates on any input state of quantum logic gate of quantum circuit, the out state remains same

3.2.3. Pauli Matrices:

Pauli matrices are the 2x 2 matrices defined by the following matrices which are very useful in the study quantum computation and quantum information.

$$\sigma_0 = I = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \qquad \sigma_X = \sigma_1 = X = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

$$\sigma_y = \sigma_2 = Y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix} \qquad \sigma_z = \sigma_3 = Z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

Most of the cases $\sigma_0 = I$ is excluded because it is generally used as identity matrix.

The operations of Pauli matrices on $|0\rangle$ and $|1\rangle$ states operations of Pauli matrices on $|0\rangle$ State :

(i) $\sigma_0 |\mathbf{0}\rangle = I |\mathbf{0}\rangle = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1+0\\0+0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = |\mathbf{0}\rangle$ $\quad I |\mathbf{0}\rangle = |\mathbf{0}\rangle$ No change

(ii) $\sigma_x |\mathbf{0}\rangle = X |\mathbf{0}\rangle = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0+0\\1+0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = |\mathbf{1}\rangle$ $\quad X |\mathbf{0}\rangle = |\mathbf{1}\rangle$

The $|\mathbf{0}\rangle$ state changed to $|\mathbf{1}\rangle$ state

(iii) $\sigma_y |\mathbf{0}\rangle = Y |\mathbf{0}\rangle = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0+0\\i+0\end{pmatrix} = \begin{pmatrix}0\\i\end{pmatrix} = i\begin{pmatrix}0\\i\end{pmatrix}$ $\quad Y |\mathbf{0}\rangle = \mathbf{i}|\mathbf{1}\rangle$

(iii) $\sigma_z |\mathbf{0}\rangle = Z |\mathbf{0}\rangle = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1+0\\0+0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ $\quad Z |\mathbf{0}\rangle = |\mathbf{0}\rangle$ No change

Operations of Pauli matrices on $|1\rangle$ state:

(i) $\sigma_0 |\mathbf{1}\rangle = I |\mathbf{1}\rangle = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0+0\\0+1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = |\mathbf{1}\rangle$ $\quad I |\mathbf{1}\rangle = |\mathbf{1}\rangle$ No change

(ii) $\sigma_x |\mathbf{1}\rangle = X |\mathbf{1}\rangle = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0+1\\0+0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = X |\mathbf{1}\rangle$ $\quad X |\mathbf{1}\rangle = |\mathbf{0}\rangle$

The $|\mathbf{1}\rangle$ state changed to $|\mathbf{0}\rangle$ state

(iii) $\sigma_y |\mathbf{1}\rangle = Y |\mathbf{1}\rangle = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0-i\\0+0\end{pmatrix} = \begin{pmatrix}-i\\0\end{pmatrix} = -i\begin{pmatrix}1\\0\end{pmatrix} = -i|\mathbf{0}\rangle$

The $|\mathbf{1}\rangle$ state changed to complex of $|\mathbf{0}\rangle$ state

(iv) $\sigma_z |\mathbf{1}\rangle = Z |\mathbf{1}\rangle = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0+0\\0-1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} = -\begin{pmatrix}0\\1\end{pmatrix} = -|\mathbf{1}\rangle$

$|\mathbf{1}\rangle$ changed to $-|\mathbf{1}\rangle$

3.2.4. Conjugate of matrix:

The complex conjugate of a given matrix is defined as that matrix whose elements are the complex conjugate of the corresponding elements of given matrix. The elements of a matrix are either real or complex

Eg: Let $A = \begin{pmatrix}1 & i\\ -i & 1\end{pmatrix}$ $\quad A^* = \begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}$

All the imaginary elements of a given matrix are replaced by opposite sign i.e., + i to – i & -i to + i

If all the elements are real, then A* = A

If all the elements are complex then A* = -A, the matrix is purely imaginary.

3.2.5. Transpose of matrix:

The transpose of matrix can be obtained by inter changing rows and columns of given matrix.

Let matrix A is defined by m x n columns A_{mn} then its transpose is given by $A^T = A_{nm}$

$$\text{Ex: } A=\begin{pmatrix}1 & 2 & 4\\ 6 & 9 & 10\\ 7 & 11 & 0\end{pmatrix} \qquad A^T=\begin{pmatrix}1 & 6 & 7\\ 2 & 9 & 11\\ 4 & 10 & 0\end{pmatrix}$$

3.2.6. Unitary Matrix:

A square matrix is said to be unitary if its inverse is equal to its conjugate transpose.

Then Unitary matrix of A is $\mathbf{A^\dagger = A^{-1} = \frac{I}{A}}$

In general $\mathbf{A^\dagger A = I = A A^\dagger}$

All unitary matrices are Hermitian

Ex:

$$A=\begin{pmatrix}cos\theta & i\sin\theta\\ i\sin\theta & \cos\theta\end{pmatrix} \qquad \mathbf{A^\dagger}=\begin{pmatrix}cos\theta & -i\sin\theta\\ -i\sin\theta & \cos\theta\end{pmatrix}$$

$$\mathbf{A^\dagger A}=\begin{pmatrix}cos\theta & i\sin\theta\\ i\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}cos\theta & -i\sin\theta\\ -i\sin\theta & \cos\theta\end{pmatrix}$$

$$\begin{pmatrix}cos^2\theta + Sin^2\theta & -i\sin\theta cos\theta + isin\theta cos\theta\\ i\sin\theta cos\theta - isin\theta cos\theta & Sin^2\theta + cos^2\theta\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=I$$

$$A^\dagger A = I$$

Note: The symbol $A^\dagger$ **(read as dagger)** represents the transpose of complex conjugate of a given matrix A. $\mathbf{A^\dagger = (A^*)^T}$

Note: The inner product or multiplication of two unitary matrixes is again unitary matrix or dot product.

3.2.7. Row and Column Matrices and their multiplication:

(i) Row matrix: A matrix has one row and n columns is called row matrix. Thus, the matrix is the order 1x n. Represented as

$$A = [\, a_{11}\ \ a_{12}\ a_{13} \ldots\ldots\ldots a_{1n}]$$

(ii) **Column Matrix:** A matrix having one column and m row is called column matrix. Thus, the matrix is of the order m x 1 represented as

$$\text{Eg:} \quad [A]=\begin{pmatrix} a_{11} \\ a_{21} \\ . \\ . \\ a_{m1} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix}_{mx\ 1}$$

The matrix multiplication (inner product):

The matrix multiplication between two given matrices exists only when **the number of columns of first matrix must be equal to number rows of second matrix**. In vector space this is also called as "**dot or inner**" product of two vectors which must be scalar quantity. The reverse multiplication

$$[A] = (1 \quad 2 \quad 0 \quad 4 \quad 5)._{1x5} \qquad [B] = \begin{pmatrix} 2 \\ 0 \\ 4 \\ 5 \\ 6 \end{pmatrix}_{5x1}$$

$$A.B = 2 + 0 + 0 + 20 + 30 = 52$$

Inner Product

Suppose, we have two states

$$|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \quad |\phi\rangle = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

One way to multiply $|\psi\rangle$ and $|\phi\rangle$ is by taking their inner product as

$$\langle\psi|\phi\rangle = (\langle\psi|)^* (|\phi\rangle)$$

$$= (\alpha_1^*, \alpha_2^*) \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$= \alpha_1^* \beta 1 + \alpha_2^* \beta_2$$

$\langle\psi|\phi\rangle$ is called the inner product and the result is always a scalar product.

3.2.8. Probability

Consider a quantum state $|\psi\rangle = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix}$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi\rangle = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

Let us take inner product with itself

$$\langle\psi|\psi\rangle = (\alpha^* \beta^*)\begin{pmatrix}\alpha\\\beta\end{pmatrix}$$

$$= \alpha\alpha^* + \beta\beta^*$$

$$= |\alpha|^2 + |\beta|^2$$

We know that $\psi\psi^* = |\psi|^2$ represents the probability

We have $\langle\psi|\psi\rangle = |\psi|^2 = |\alpha|^2 + |\beta|^2 = 1$

This implies $|\psi\rangle$ is normalized.

3.2.9. Orthogonality

Consider the inner product of $|0\rangle$ and $|1\rangle$

$$\langle 0|1\rangle = (1\ 1)\begin{pmatrix}0\\1\end{pmatrix}$$

$$= 1.0+0.1$$

$$= 0$$

If the inner product of two abstract state vectors is zero, then they are said to be orthogonal.

i,e. $\langle 0|1\rangle = 0$

3.2.10. Orthonormality

Two states $|\psi\rangle$ and $|\phi\rangle$ are said to be orthonormal if

(1) $|\psi\rangle$ and $|\phi\rangle$ normalized

(2) $|\psi\rangle$ and $|\phi\rangle$ are orthogonal to each other.

Quantum Superposition:

Quantum superposition is the ability of a quantum system to exist in multiple states at the same time until it is measured.

Let us consider Dirac notation of quantum system

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ which are state of quantum mechanical system or physical object which are called quantum bits or Qubits. The qubits exist not only at $|0\rangle$ and $|1\rangle$ but in the states of linear combination also, often called "Superposition".

i.e, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

where α and β are complex numbers, $|\psi\rangle$ is a state of quantum system or wavefunction.

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3.3 Quantum Gates

The basic purpose of a logic gate is to manipulate or process information at the bit level.

Single Qubit Gates:

Consider a unitary operator U represent operation on 2-dimensional vector space of a single qubit as 2X2 matrix operators on a single qubit (Unitary) gate. A linear operator is specified completely by its action on the basis vectors.

Every quantum gate is a unitary operator which satisfies the conditions

(i) $U^{\dagger}\ U = U\ U^{\dagger} = I$

(ii) $|\det| = 1$

(iii) Columns satisfy orthonormal condition. i,e. self-product is equal to 1 and different product is equal to 0.

3.3.1. Quantum NOT Gate:

A quantum NOT gate is one which map $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$.

Being a linear operator, it will map a linear combination of inputs to the corresponding linear combination of output. Hence the NOT gate maps the general state as

$$\alpha_1|0\rangle + \alpha_2|1\rangle \rightarrow \boxed{\text{NOT operation}} \rightarrow \alpha_1|1\rangle + \alpha_2|0\rangle$$

Input Output

In terms of the matrix representation, the

NOT operation on $\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

NOT operation on $\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$

From the above, the matrix of NOT gate can be obtained or constructed in the computation as basis by 2x2 matrix as $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

The quantum NOT or QUNOT gate can be represented as

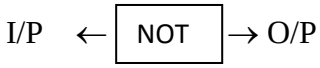


Fig. quantum NOT gate

Table. Truth table of NOT gate:

Input	**output**
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\alpha\|0\rangle + \beta\|1\rangle$	$\alpha\|1\rangle + \beta\|0\rangle$

3.3.2. Pauli –X, Y and Z gates:

Pauli X-gate:

When Pauli X gate operates two states $|0\rangle$ and $|1\rangle$ which can be written in outer product form as follows

i)Leave zero products

ii)All bra vectors changes to ket and ket vectors changed to bra vectors

iii)Follow orthonormal conditions for bra and ket state vector i.e.,

$$\langle 0|0\rangle = 1, \langle 1|1\rangle = 1$$

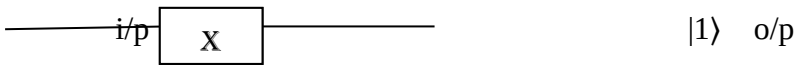
$$\langle 0|1\rangle = 1, \langle 1|0\rangle = 0$$

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Inner Product:

$$\mathbf{X}\,|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0+0 \\ 1+0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$$\mathbf{X}\,|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0+0 \\ 1+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |0\rangle$$



Pauli X gate when operates on $|0\rangle$ flip to $|1\rangle$ and $|1\rangle$ to $|0\rangle$, where two states are inverted.

Bloch sphere representations

Pauli -Y gate:

Consider Pauli Y gate

$Y=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$

Inner product:

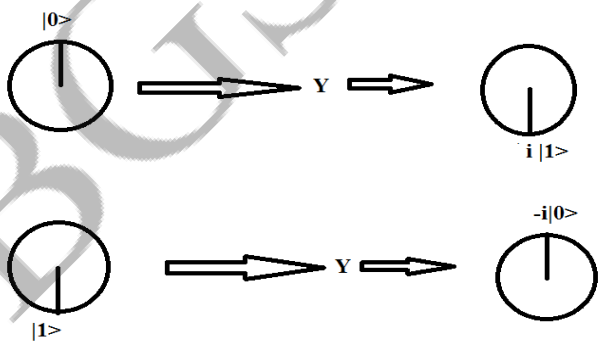
$Y\,|0\rangle = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0+0\\ i+0\end{pmatrix} = \begin{pmatrix}0\\ i\end{pmatrix} = i|1\rangle$

$Y\,|0\rangle = i|1\rangle$---------- (3)

$Y\,|1\rangle = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0-i\\ 0+0\end{pmatrix} = \begin{pmatrix}-i\\ 0\end{pmatrix} = -i\begin{pmatrix}1\\0\end{pmatrix} = i|0\rangle$

$Y\,|1\rangle = -i|0\rangle$----------(4)

Block sphere representations:



Z-Gate:

The Z gate matrix is

$\mathbf{Z} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

Consider a Z gate operation on $|0\rangle$ & $|1\rangle$ state. Now the outer product form operation of Z gate.

Inner product:

It leaves |0⟩ unchanged and inverts the sign of |1⟩ to give -|1⟩.

For |0⟩, the output becomes

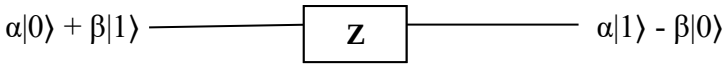
$Z\,|0\rangle = |0\rangle$

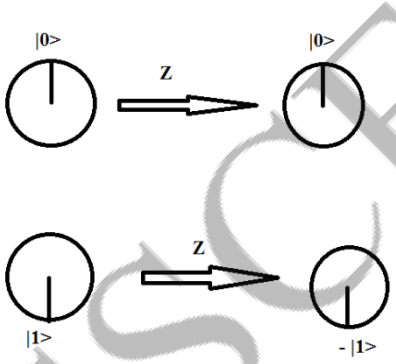
Or $\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} = |0\rangle$

In case of |1⟩, the output is

$Z\,|1\rangle = -|1\rangle$

Or $\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix} = -|1\rangle$





3.3.3. Hadamard Gate: Hadamard gate is also called H-gate. The Hadamard gate is a type of single qubit quantum gate a basic operation that can be applied to a qubit. Like all quantum gates, the Hadamard gate is a unitary transformation on a vector space.

The Hadamard gate is defined as

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}$$

Inner Product:

$$H|0\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}+0\\ \frac{1}{\sqrt{2}}+0\end{pmatrix}$$

$$H|0\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

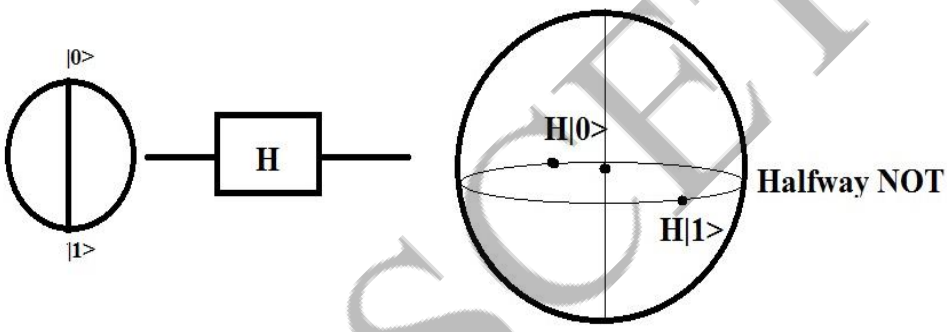
$$H|0\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} + 0 \\ \frac{-1}{\sqrt{2}} + 0 \end{pmatrix}$$

$$H|0\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right]$$

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Hence Hadamard hate is a linear combination or summation of Pauli X and Z gate

$H^2 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



$H = \sqrt{NOT}$ and $NOT \neq H^2$

3.3.4. Phase gate (or) S gate

The Phase gate or S gate is a single qubit operation defined by,

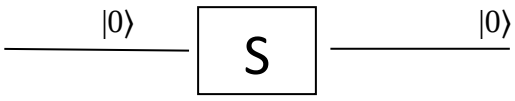
$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix}$

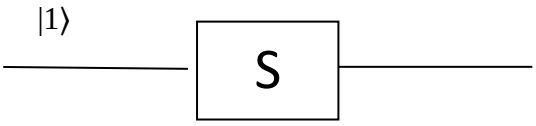
The S gate is also known as the Phase gate or Z 90˚ gate, because it represents a 90˚-degree rotation around the Z axis

Inner Product:

$$\mathbf{S}|\mathbf{0}\rangle = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+0 \\ 0+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\mathbf{0}\rangle$$

$$\mathbf{S}|\mathbf{1}\rangle = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0+0 \\ 0+\mathrm{i} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{i} \end{pmatrix} = \boldsymbol{i}|\mathbf{1}\rangle$$





3.3.5. T gate

The T gate is a single qubit operation defined by,

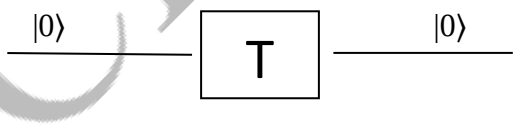
$$S = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

The S gate is also known as the π/8 gate

Inner Product:

$$T|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+0 \\ 0+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$T|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0+0 \\ 0+e^{i\pi/4} \end{pmatrix} = \begin{pmatrix} 0 \\ e^{i\pi/4} \end{pmatrix} = e^{i\pi/4} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{i\pi/4}|1\rangle$$



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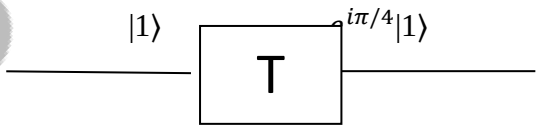


Cross Product (Tensor Product):

$$|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \qquad |\Phi\rangle = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

$$|\psi\rangle \otimes |\Phi\rangle = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_2 & \beta_2 \end{pmatrix}$$

Ex: $|0\,0\rangle = |0\rangle \otimes |\Phi\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

3.3.6. Multiqubit gate:

Quantum gates are unitary operators

Two qubit quantum states:

The basis states of two qubits quantum gates are

$$\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$$

$$|0\ 0\rangle \quad , |0\ 1\rangle \quad , |1\ 0\rangle \quad , |1\ 1\rangle$$

$$|00\rangle \ , |01\rangle \ , |10\rangle \ , |11\rangle$$

$$|00\rangle = 0 \text{ X } 2^1 + 0 \text{ X } 2^0 = 0$$

$$|01\rangle = 0 \text{ X } 2^1 + 1 \text{X } 2^0 = 1$$

$$|01\rangle = 1 \text{ X } 2^1 + 0 \text{X } 2^0 = 2$$

$$|11\rangle = 1 \text{ X } 2^1 + 1 \text{X } 2^0 = 3$$

$$\{|00\rangle \ , |01\rangle \ , |10\rangle \ , |11\rangle\} = \{0,1,2,3\}$$

Two qubit quantum gates also represented as above equation 1,

Outer Product of basis states is given by

$$|\mathbf{0}\rangle \mathbf{x} |\mathbf{0}\rangle = \begin{pmatrix}1\\0\end{pmatrix} X \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$|10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Take column matrix of all above basis state write a column of matrix we will get,

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} = I_4$$

Three Qubit Quantum states:

Tensor Product of basis is given by

The basis states of three qubit quantum gates are,

$|0\rangle \otimes |0\rangle \otimes |0\rangle$ = n as $|000\rangle$

All the basis states of three qubit quantum logic gates can be written as,

$|000\rangle$, $|001\rangle$,$|010\rangle$,$|011$,$|100\rangle$,$|101\rangle$,$|110\rangle$ $|111\rangle$

0 1 2 3 4 5 6 7

(0 to 2^3-1)

$$|000\rangle = \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} \quad |001\rangle = \begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix} \quad |010\rangle = \begin{pmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix} \quad |011\rangle = \begin{pmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{pmatrix}$$

$$|100\rangle = \begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{pmatrix} \quad |101\rangle = \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{pmatrix} \quad |110\rangle = \begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{pmatrix} \quad |111\rangle = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}$$

$$I_8 = \begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{pmatrix}$$

3.3.7. Controlled Gate:

In case of controlled gate "If A is true then do B". A is generally known as the **control qubit** and B as the **target qubit**.

- If the control qubit is 0, then the target qubit is not altered.
- If the control qubit is 1, then the target qubit is transformed.
- However, the control qubit remains unaltered in both the cases.

3.3.8. Controlled NOT (CNOT Gate)

The prototypical multi-qubit quantum logic gate is the controlled NOT or CNOT gate. IT has two input qubits:

(i) **Control qubit**: This is represented by the top line in the circuit given below.

(ii) Target qubit: This is shown by the bottom line of the figure.

CNOT gate

If the control qubit (A) of the gate is set to 0, then the target qubit (B) is not altered. IF the control qubit is set to 1, then the target qubit is inverted.

(a) Input state |00⟩ (Control qubit = 0, Target qubit = 0): Both the bits remain unaltered. Hence, the output state is the same as the input state or

$$|00\rangle \rightarrow |00\rangle$$

(b) **Input state |01⟩ (Control qubit = 0, Target qubit = 1):** Both the bits remain unaltered. Hence, the output state is the same as the input state or

$$|01\rangle \rightarrow |01\rangle$$

(c) **Input state |10⟩ (Control qubit = 1, Target qubit = 0):** The target qubit is flipped to 1. Therefore, the output state has both qubits 1, or

$$|10\rangle \rightarrow |11\rangle$$

(d) **Input state |11⟩ (Control qubit = 1, Target qubit = 1):** The target qubit is flipped to 0. Therefore, the output state becomes |10⟩, or

$$|11\rangle \rightarrow |10\rangle$$

The truth table of a CNOT gate is given below.

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

It can be represented in the matrix form as,

$$U_{CN}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

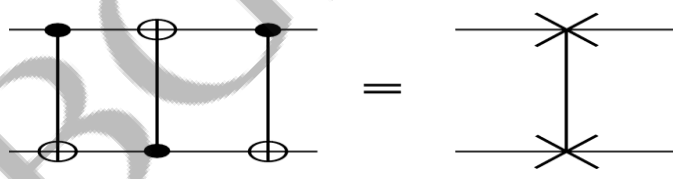
The CNOT gate can be regarded as a type of generalized XOR gate since the action of the gate can be considered as

$$|A, B\rangle \rightarrow |A, B\oplus A\rangle$$

Where $\oplus$ stands for modulo-2 addition, which is the same as that achieved by a XOR gate. In other words, the control qubit and the target qubit are XORed and stored in the target qubit.

3.3.9. Swap Gate:

The Swap Gate is a simple quantum circuit containing 3 quantum gates. The circuit is read from left to right and each line indicates a quantum passage, maybe of time, perhaps a physical particle like a photon or a particle of light to move from one location to another space.



Swap gate

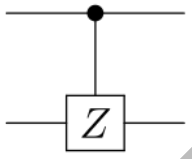
The output of the first CNOT gate is $|a, a \oplus b\rangle$. This is fed as input to the second gate which also does modulo 2 additions but the result is placed in the first qubit now. Its output becomes $|a\oplus (a\oplus b), a\oplus b\rangle = |b, a\oplus b\rangle$.This is now fed to the third gate which performs modulo 2 addition and places the result in the second qubit. The output finally becomes $|b, (a\oplus b) \oplus b) = |b, a\rangle$. Thus, the overall effect is that the two qubits have been swapped.

The truth table of a Swap gate is given below.

Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|10\rangle$
$\|10\rangle$	$\|01\rangle$
$\|11\rangle$	$\|11\rangle$

3.3.10. Controlled- Z gate

The Controlled-Z gate includes the operation of the kind "If A is true, then do B". Below figure shows a Controlled-Z gate.



Controlled-Z gate

$$U_z = \begin{pmatrix} 1 & & & \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\mathbf{1} \end{pmatrix}$$

Truth table of a Controlled-Z gate

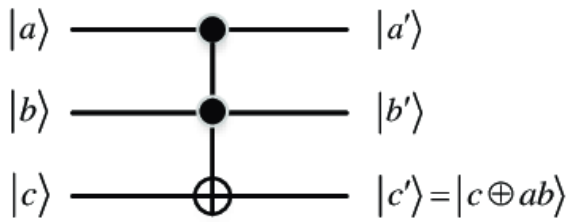
Input	Output
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|10\rangle$
$\|\mathbf{11}\rangle$	$-\|\mathbf{11}\rangle$

3.3.11. Toffoli Gate

Any classical logic circuit can be built using a quantum circuit. Physicists believe that all aspects of the world around us including classical logic circuits, can ultimately be explained using quantum mechanics. Any classical circuit can be replaced by an equivalent circuit containing only reversible elements, by using a reversible gate called Toffoli Gate. Below figure shows a Toffoli gate and the truth table.

Inputs			Ouputs		
a	b	c	a'	b'	c'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

Truth table



Toffoli gate

The Toffoli gate has three input bits (a, b, c) and three output bits (a', b', c'). The first two bits are control bits which remain unaffected by the action of the Toffoli gate. The third is the target bit which is inverted if both the control bits are 1, else it is left unchanged.

The Toffoli gate can be expressed as 8 x 8 matrix

$$I_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \end{pmatrix}$$

Numerical Problems:

Problem 1:

Consider A be the linear operator from V to V such that A $|\mathbf{0}\rangle = |\mathbf{1}\rangle$ and A $|\mathbf{1}\rangle = |\mathbf{0}\rangle$ **.**Given matrix representation of A.

Let A = $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ and We know that $|\mathbf{0}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $|\mathbf{1}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Now

$$A|\mathbf{0}\rangle = |\mathbf{1}\rangle$$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_{11} + 0 \\ x_{21} + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x_{11} = 0 \qquad\qquad x_{21} = 1$$

$$A|\mathbf{1}\rangle = |\mathbf{0}\rangle$$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0+ & x_{12} \\ 0+ & x_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$x_{12} = 1$ $\qquad\qquad$ $x_{22} = 0$

Hence the linear operator defined by its matrix elements

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Problem 2:

Show that the matrix A = $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix}$ is unitary

$$\mathbf{A}^{\dagger} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$\mathbf{A}^{\dagger}\,\mathbf{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}+\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} & \frac{1}{2}+\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\mathbf{A}^{\dagger}\,\mathbf{A} = I$$

Problem 3:

Find the Inner product of states $|0\rangle$ & $|1\rangle$ and draw conclusions on the result:

Solution:

We know that $|\mathbf{0}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad\qquad$ $|\mathbf{1}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\langle 0| = (0 \quad 1)$$

The inner product is given by,

$$\langle 1|0\rangle = (0 \quad 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (0+0) = 0$$

Thus, the states are orthogonal.

Problem 4:

Given A = $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ Prove that $\mathbf{A}^{\dagger} = \mathbf{A}$

Solution: A = $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ A* = A = $\begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix}$

$\mathbf{A}^{\dagger} = (\mathbf{A}^{*})^{\mathbf{T}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = A$

Hence $\mathbf{A}^{\dagger} = \mathbf{A}$

Problem 5:

$|\psi\rangle$ = A [2$|0\rangle$ + 3i$|1\rangle$ find $\langle\psi|\psi\rangle$

Solution: $|\psi\rangle = A[\ 2\begin{pmatrix}1\\0\end{pmatrix} + 3i\begin{pmatrix}0\\1\end{pmatrix} = A\ [\begin{pmatrix}2\\0\end{pmatrix} + \begin{pmatrix}0\\3i\end{pmatrix}] = A[\begin{pmatrix}2\\3i\end{pmatrix}]$

$\langle\psi| = A\ [\ 2 \quad 3i]$

$\langle\psi|\psi\rangle = A\ [\ 2 \quad -3i]\ .A[\begin{pmatrix}2\\3i\end{pmatrix}]$

$\langle\psi|\psi\rangle = A^2\ [4+9] = A^2\ .[13]$

$|\psi|^2 = A^2\ .[13]$

$1 = A^2\ .[13]$

$A = 1/\sqrt{13}$

Problem 6:

Check the orthogonality $|i\rangle$ = $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ $|-i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

$\langle i|-i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}.\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

$= \frac{1}{2}(1 + i^2) = \frac{1}{2}(1 - 1) = 0$

Problem 7:

Show that (1,-1), (1,2) & (2,1) are linearly dependent

Solution:

$\alpha_1\begin{pmatrix}1\\-1\end{pmatrix} + \alpha_2\begin{pmatrix}1\\2\end{pmatrix} + \alpha_3\begin{pmatrix}2\\1\end{pmatrix}$

$\alpha_1|v_1\rangle + \alpha_2|v_2\rangle + \alpha_3|v_3\rangle = 0$

$\alpha_1 + \alpha_2 + 2\alpha_3 = 0$ ----------(1)

$-\alpha_1 + 2\alpha_2 + \alpha_3 = 0$

$3\alpha_2 + 3\alpha_3 = 0$

$\alpha_2 = -\alpha_3$ ---------- (2)

Equation (2) in (1)

⇨ $\alpha_1 - \alpha_3 + 2\alpha_3 = \mathbf{0}$

$\alpha_1 + \alpha_3 = \mathbf{0} \Rightarrow \alpha_1 = -\alpha_3$

Put $\alpha_3 = 1 \Rightarrow \alpha_2 = -1$

$\alpha_1 = -1$

$\alpha_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0$

$\Rightarrow (-1) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (-1) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (1) \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0$

$\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0$

-1-1+2 =0 hence they are linearly dependent.

1-2+1=0

Problem:8

Given $|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ $|\Phi\rangle = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ Prove that $\langle\psi|\Phi\rangle = \langle\Phi|\psi\rangle^*$

Solution: $|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ $\langle\psi| = (\alpha_1^* \ \alpha_2^*)$

$|\Phi\rangle = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ $\langle\Phi| = (\beta_1^* \ \beta_2^*)$

$\langle\psi|\Phi\rangle = (\alpha_1^* \ \alpha_2^*) \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \alpha_1^* \beta_1 + \alpha_2^* \beta_2$ ----------(1)

$\langle\Phi|\psi\rangle = (\beta_1^* \ \beta_2^*) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \alpha_1 \beta_1^* + \alpha_2 \beta_2^*$ ----------(2)

$\langle\psi|\Phi\rangle^* = \langle\psi^*|\Phi^*\rangle$

$= \langle\psi^*| = |\psi\rangle$, $\langle\Phi^*| = |\Phi\rangle$

$= \langle\Phi|\psi\rangle$

$\langle\psi|\Phi\rangle^* = (\beta_1^* \ \beta_2^*)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \alpha_1 \ \beta_1^* + \alpha_2 \ \beta_2^*$ ---------- (3)

From equ .2 and 3

$\langle\psi|\Phi\rangle = \langle\Phi|\psi\rangle^*$

Problem 9: Show that Hadamard gate is Unitary

Solution:

Consider H = $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ unitary operator means $\mathbf{H^{\dagger}\cdot H = I}$

$\mathbf{H^{\dagger} = (H^{*})^{T}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ = H

$\mathbf{H^{\dagger}\cdot H} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}.\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix}1+1 & 1-1\\ 1-1 & 1+1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$\mathbf{H^{\dagger}\cdot H = I}$

Problem 10:

Show that S gate can be formed by connecting two T gate in series

Solution:

The S gate is also known as the π/8 gate

$T = \begin{pmatrix}1 & 0\\ 1 & e^{i\pi/4}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 1 & e^{\left(\frac{i\pi}{2}\right)^{1/2}}\end{pmatrix}$

$= \begin{pmatrix}1 & 0\\ 1 & e^{\left(\frac{i\pi}{2}\right)}\end{pmatrix}^{1/2} = \sqrt{S}$

T= $\sqrt{S}$ or $T^2 = S$

Problem 11:

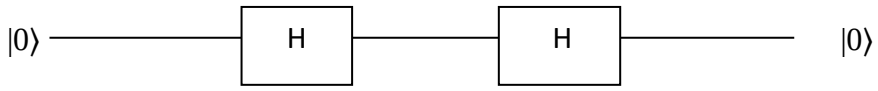
Using matrix multiplication show that on applying Hadamard gate twice to a |0⟩ results in its original gate.

$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

$H.H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}.\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1+1 & 1-1 \\ 1-1 & 1+1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

H.H =I when H^2 operates on $|0\rangle$ or I operates $|0\rangle$the resultant will be original Qu gate.



Problem 12: Using two gates that two N0T gates in series are equivalent to a quantum wire.

Solution

Two Qu gates NOT operation on basis state $|0\rangle$ and $|1\rangle$ from a quantum wire

Hence show that,

$X.X|0\rangle = |0\rangle$ or $X.X = I$

$\mathbf{X} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $X.X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$X.X|0\rangle = |0\rangle$ or $X.X = I$

Hence two N0T gates in series are equivalent to a quantum wire.

Module-4

Electrical Properties of Materials and Applications

Electrical Conductivity in metals

Resistivity and Mobility, Concept of Phonon, Matheissen's rule, Failures of Classical Free Electron Theory, Assumptions of Quantum Free Electron Theory, Fermi Energy, Density of States, Fermi Factor, Variation of Fermi Factor with Temperature and Energy. Numerical Problems.

Superconductivity

Introduction to Super Conductors, Temperature dependence of resistivity, Meissner's Effect, Critical Field, Temperature dependence of Critical field, Types of Super Conductors, BCS theory (Qualitative), Quantum Tunnelling, High Temperature superconductivity, Josephson Junctions (Qualitative), DC and RF SQUIDs (Qualitative), Applications in Quantum Computing: Charge, Phase and Flux qubits, Numerical Problems.

4.1. Electrical Conductivity in metals

4.1.1. Resistivity (ρ):

Electrical resistivity is **a measure of how strongly it opposes the flow of current**. The ohm meter (m) is the SI unit of electrical resistivity.

"The electrical resistivity of a material of a conductor is defined as the resistance offered by the unit length and unit cross-sectional area of a wire of the given material".

MOBILITY:

"**It is defined as the magnitude of the drift velocity v_d acquired by the electrons in an electric field E".**

$$\mu = \frac{v_d}{E}$$

Where μ is called the mobility of the charge carrier.

CONDUCTIVITY (σ): It characterizes the conducting capability of the material. It is denoted by σ and it is inversely proportional to the resistivity.

$$\sigma = \frac{1}{\rho}$$

Electrical Conductivity:

It is defined as the rate of charge flow across unit area in a conductor. Electrical conductivity (σ) per unit potential (voltage) gradient.

$$\sigma = \frac{J}{E}$$ Its unit is Ω^{-1} m $^{-1}$

(a) **CURRENT DENSITY (J):** If 'I' is the current flowing in a conductor of cross sectional area "A" then the current density 'J' is the current flowing per unit area of cross- section of an imaginary plane held normal to the direction of current in a current carrying conductor.

$$J = \frac{I}{A}$$

(b) **ELECTRIC FIELD(E):** If 'L' is the length of the conductor of uniform cross sectional area "A" and of uniform material composition, and 'V' is the potential difference between the two ends then the electric field 'E' at any point inside is given by

$$E = \frac{V}{L}$$

Note: Some important Definitions:

1. Thermal Velocity (vth): The velocity with which the free electrons keep moving due to thermal agitation is called Thermal velocity.

2. Drift velocity (vd): The velocity of electrons in the steady state in an applied electric field is called drift velocity.

3. Mean free path (λ): The average distance travelled by the conduction electrons between successive collisions with lattice ions.

4. Mean collision time *(τ)*: The average time that elapses between two consecutive collisions of an electron with the lattice points is called mean collision time.

$$\tau = \lambda/v$$

where 'λ' is the mean free path, $v \approx v_{th}$ is velocity same as combined effect of thermal & drift velocities.

5. Relaxation time (τ_r): From the instant of sudden disappearance of an electric field across a metal, the average velocity of the conduction electrons decays exponentially to zero, and the time required in this process for the average velocity to reduce to (1/e) times its value is known as Relaxation time.

6. Electrical conductivity σ = $\frac{ne^2\lambda}{\sqrt{3kTm}}$ (according to classical free electron theory)

Electrical Conductivity **σ** = $\frac{ne^2\tau}{m}$ (in terms of mean collision)

4.1.2. Concept of phonon:

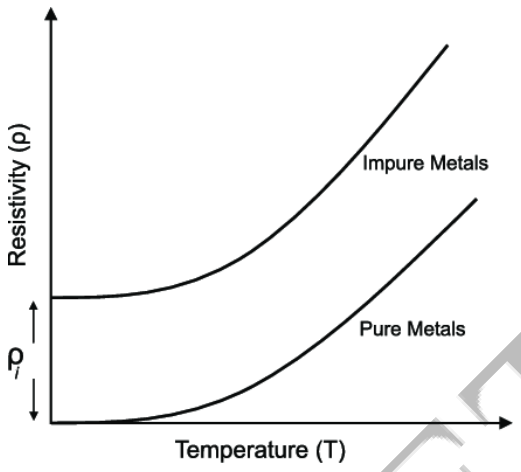
"Phonons are quantized sounds that are created due to the oscillation of atoms within a crystal. It is a form of vibrational energy"

All metals and their alloys are crystalline materials. Hence the atoms or ions are situated at a particular position in a crystal lattice. Because of the thermal energy these ions are vibrating in all directions from the mean position in space. These vibrated ions in crystal lattice are quantized. The quantized lattice vibrations are called phonons. The amplitude of lattice vibrations directly proportional to temperature.

Variation of resistivity with temperature and Impurity: (Mathiessen's Rule)

The resistivity of metals to the flow of current is due to scattering of the conduction electrons by lattice vibrations or phonon. Hence the electron moves under the action of electric field in the phonon field. The amplitude of lattice vibrations or phonons proportional to temperature.

If the temperature of the material increases amplitude of the lattice vibration also increases, then more and more electrons scattered which leads to increases the resistivity of metals directly proportional to temperature. The dependence of resistivity of metals under normal conducting state is shown in fig. below



4.1.3. Mathiessen's Rule:

"The total resistivity of a metal is the sum of the resistivity due to phonon scattering which is temperature dependent and the resistivity due to scattering by impurities which is temperature independent"

$$\boldsymbol{\rho = \rho_{ph} + \rho_i}$$

Where $\rho_{ph}\ is$ the resistivity due to lattice vibration is increases with temperature and ρ_i is the resistivity due to scattering of conduction electrons by impurities and imperfections.

4.1.4. Classical free electron theory:

The properties of metals such as electrical conduction, thermal conduction, specific heat etc are due to the free electrons or conduction electrons in metals. The first theory to explain the electrical conductivity of metals is 'Free electron theory' and it was proposed by Drude in the year 1900 and later developed and refined by Lorentz. Hence the classical free electron theory is known as Drude- Lorentz's theory. It failed to account the facts such as temperature dependence of conductivity and dependence of electrical conductivity on electron concentration.

Failures of Classical Free Electron Theory:

Electrical and thermal conductivities can be explained from classical free electron theory. It fails to account the facts such as specific heat, temperature dependence of conductivity and dependence of electrical conductivity on electron concentration.

1. Temperature dependence of electrical conductivity:

We know, the conductivity of a metal is proportional to inverse of temperature, i.e.,

$$\sigma = \frac{1}{T}$$

But following Drude-Lorentz theory one arrives at the condition $\sigma = \frac{1}{\sqrt{T}}$ which is not valid according to quantum mechanics.

2. Dependence of electrical conductivity on electron concentration:

According to classical free electron theory, electrical conductivity is given by,

$$\sigma = \frac{ne^2\lambda}{\sqrt{3kTm}}$$

therefore, σ α n Experimentally determined values of some metals are given below

Metal	**Electron Concentration (n) 10^{28}/ m^3**	**Electrical Conductivity (σ) 10^7 /Ωm**
Copper	8.45	5.88
Aluminum	18.46	3.66

From the table it is clear that, though electron concentration in copper is 2.13 times less than that of Aluminum, its electrical conductivity is much greater than that of Al. Hence, the classical free electron theory fails to explain the dependence of σ on electron concentration.

3. Specific heat of conduction electrons:

The specific heat of conduction electrons as per classical free electron theory is given by

$$C_v = 3/2\ R$$

Where R is gas constant and C_v is the specific heat of conduction electrons. But experimental evidence showed that the C_v of conduction electron is

$$C_v = 10^{-4}\ RT$$

Where Cv is the specific heat, R is a gas constant, T is absolute temperature of metal

From the above discussions, it is clear that classical free electron theory fails to explain many of the experimentally observed facts.

4.1.5. Quantum Free Electron Theory

In order to rectify the draw backs of the classical free electron theory, Sommerfeld proposed the quantum free electron theory in 1928. This theory is based on quantum concepts.

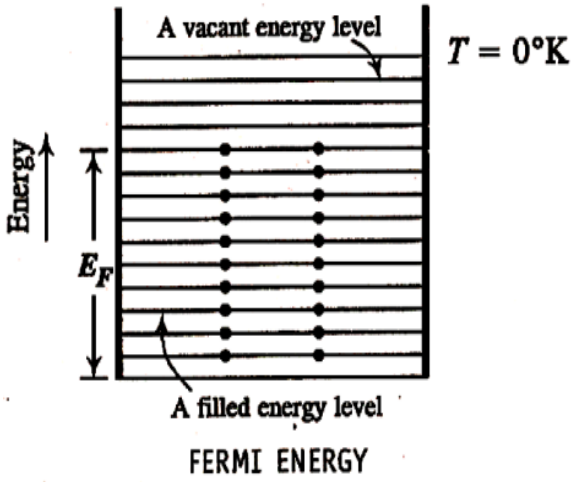
Assumptions of quantum free electron theory.

1. The energy values of the conduction electrons are quantized.
2. The distribution of energy among the free electrons is according to Fermi-Dirac statistics.
3. The distribution of electrons in the various allowed energy levels occur as per Pauli's exclusion principle.
4. The electrons travel with a constant potential inside the metal but confined within its boundaries.

4.1.6. Fermi Energy

In a metal having N atoms, there are N allowed energy levels in each band. In the energy band the energy levels are separated by energy differences. It is characteristic of the material. According to Pauli's exclusion principle, each energy level can accommodate a maximum of two electrons, one with spin up and the other with spin down. The filling up of energy levels occurs from the lowest level. The next pair of electrons occupies the next energy level and so on till all the electrons in the metal are accommodated. Still number of allowed energy levels, are left vacant.

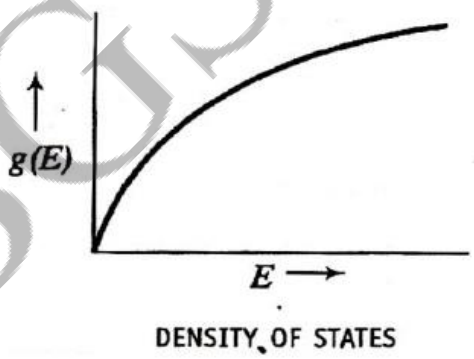
"The energy of the highest occupied level at absolute zero temperature (0K) is called the *Fermi energy* and the energy level is called *Fermi level'*. It is denoted by Ef "



FERMI ENERGY

4.1.7. Density of States:

There are large numbers of allowed energy levels for electrons in solid materials. A group of energy levels close to each other is called as energy band. Each energy band is spread over a few electron-volt energy ranges. In $1mm^3$ volume of the material, there will be large number of permitted energy levels in an energy range of few electron-volts. Because of this, the energy values appear to be virtually continuous over a band spread. To represent it technically it is stated as density of energy levels. The dependence of density of energy levels on the energy is denoted by g(E). It is called density of states function. The graph shows variation of g(E) versus E.



DENSITY OF STATES

It is the number of allowed energy levels per unit energy interval in the band associated with material of unit volume'.

The density of states in range E and (E+dE) is denoted by g(E)dE.

$$g(E)dE = \frac{8\sqrt{2}\pi m^{3/2}}{h^3} E^{1/2}\, dE$$

It is clear g(E) is proportional to $\sqrt{E}$ in the interval dE.

4.1.8. Fermi Factor:

When the temperature is greater than zero Kelvin, the material will receive thermal energy from its surroundings at room temperature. The available thermal energy is small and therefore the electrons occupying energy levels much below the Fermi level cannot absorb the thermal energy due to non-availability of higher energy levels. But this energy can be absorbed by the electrons occupying energy levels which are just below the Fermi level. Because there are a large number of unoccupied energy levels just above the Fermi level to which electrons get excited. Though the excitations are random, the distributions of electrons in various energy levels will be systematically governed by a statistical function at the steady state.

'*Fermi factor* is defined as the probability of occupancy of a given energy state for a material in thermal equilibrium'.

The probability f (E) that a given energy state with energy E is occupied at a steady temperature T is given by

$$f(E)= \frac{1}{1+e^{\frac{E-E_f}{KT}}}$$

f (E) is called the Fermi factor

Where, E_F - Fermi energy (highest energy level of an electron)

K - Boltzmann's constant

T - Absolute temperature

4.1.9. Dependence of fermi factor with temperature and energy:

The dependence of Fermi factor on temperature and energy is as shown in the figure.

Case I: Probability of occupancy for E< E_f at T=0K:

When T= 0K and E<Ef

$$f(E)= \frac{1}{1+e^{-\infty}} = \frac{1}{0+1} = 1$$

Therefore, at T=0K all the energy levels below the Fermi level are occupied.

Case II: Probability of occupancy for E>Ef at T=0K:

When T=0K and E >Ef

$$f(E)= \frac{1}{1+e^{\infty}} = \frac{1}{\infty} = 0$$

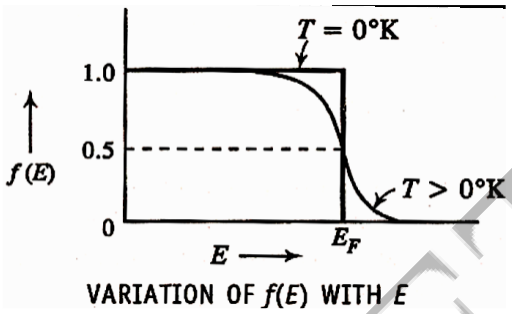
f (E)=0 for E >Ef

... At T=0K, all the energy levels above fermi levels are unoccupied

Case III: The probability of occupancy at T>0K

At E=E_f, $^{(E-Ef)/kT} = e^0 = 1$

f(E)= f(E)= $\frac{1}{1+e^{\frac{E-E_f}{KT}}} = \frac{1}{1+1} = \frac{1}{2} = 0.5$



VARIATION OF $f(E)$ WITH E

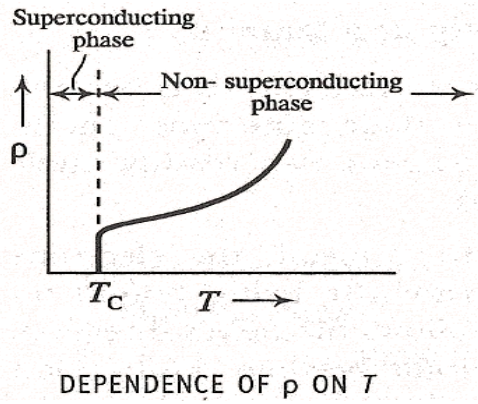
Hence, the Fermi energy is the most probable or the average energy of the electrons across which the energy transitions occur at temperature above zero-degree absolute.

4.2. SUPERCONDUCTIVITY

Superconductivity is the phenomenon observed in some metals and materials. Kammerlingh Onnes in 1911 observed that the electrical resistivity of pure mercury drops abruptly to zero at about 4.2K. This state is called super conducting state. The material is called superconductor. The temperature at which they attain superconductivity is called critical temperature T_c.

4.2.1. Temperature dependence of resistivity of a superconductor:

One of the most interesting properties of solid at low temperature is that electrical resistivity of metals and alloys vanish entirely below a certain temperature. This zero resistivity or infinite conductivity is known as superconductivity. Temperature at which transition takes place is known as *transition temperature or critical temperature (T_c).* Above the transition temperature, the substance is in the normal state and below it will be in superconducting state. T_c value is different for different materials.

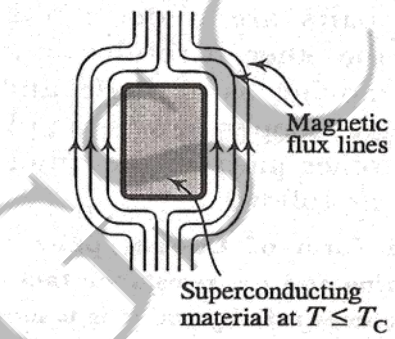


DEPENDENCE OF ρ ON T

"The resistance offered by certain materials to the flow of electric current abruptly drop to zero below a threshold temperature. This phenomenon is called *superconductivity* and threshold temperature is called "*critical temperature.*"

4.2.2. Meissner effect

A superconducting material kept in a magnetic field expels the magnetic flux out of its body when it is cooled below the critical temperature and thus becomes perfect diamagnet. This effect is called Meissner effect.



EXPULSION OF MAGNETIC FLUX

When the temperature is lowered to T_c, the flux is suddenly and completely expelled, as the specimen becomes superconducting. The Meissner effect is reversible. When the temperature is raised the flux penetrates the material, after it reaches T_c. Then the substance will be in the normal state.

The magnetic induction inside the specimen

$$B = \mu_o (H + M)$$

Where 'H' is the intensity of the magnetizing field and 'M' is the magnetization produced within the material.

$$\text{For } T < T_c, \qquad B = 0$$

$$\mu_0 (H + M) = 0$$

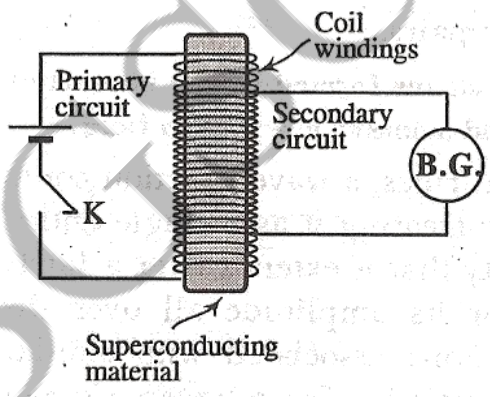
$$M = -H$$

$$\frac{M}{H} = -1 = \chi$$

Susceptibility is -1 i.e. it is perfect diamagnetism.

Hence superconducting material do not allow the magnetic flux to exist inside the material.

Consider a primary coil and a secondary coil, wound on a superconducting material. The primary coil is connected to a battery and a key. The secondary coil is connected to ballistic galvanometer (BG). When the key is closed the current flows through the primary coil and the magnetic field is produced. This flux is linked with the secondary coil and the current flows through the secondary coil which makes a deflection in the galvanometer. If the primary current is steady the magnetic flux and the flux linked with the coil will become steady. As the temperature of the specimen is decreased below the critical temperature, BG suddenly shows a deflection indicating that the flux linked with the secondary coil is changed. This is due to the expulsion of the magnetic flux from the specimen.



MEISSNER EFFECT

4.2.3. Temperature dependence of Critical field:

Critical temperature (T_C) (or) Transition Temperature :

The temperature at which a normal conductor loses its resistivity and becomes a superconductor is known as critical temperature (or) Transition temperature. Every superconductor has its own critical temperature at which it passes over into superconducting state. Depending on the transition temperature, superconductors are classified into two groups are

i) **Low temperature superconductors (LTS):**The superconductors which have low transition temperature (below 30K) are known as low temperature superconductors. Example: Tin (3.2 K), Mercury (4.15 K).

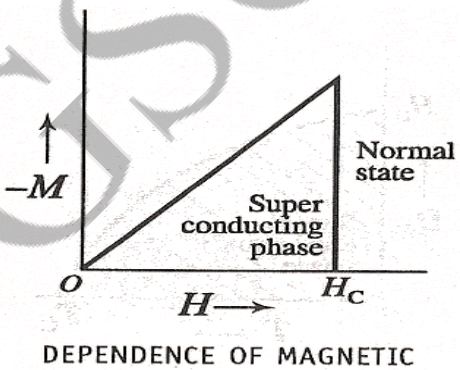
ii) High temperature superconductors (HTS): The superconductors which have high transition temperature (above 30K) is known as high temperature superconductors. Example: Barium - Lanthanum - Copper - Oxide (BLCO) - 35 K Yttrium - Barium - Copper - Oxide - (Y Ba2 Cu3 O4) - 92 K

4.2.4. Types of superconductors

There are two types of superconductors. They are type-I superconductors and type-II superconductors.

Type-I superconductors:

Type-I superconductors exhibit complete Meissner effect. Below the critical field it behaves as perfect diamagnet. If the external magnetic field increases beyond H_c the superconducting specimen gets converted to normal state. The magnetic flux penetrates and resistance increase from zero to some value. As the critical field is very low for type-I superconductors, they are not used in construction of solenoids and superconducting magnets.

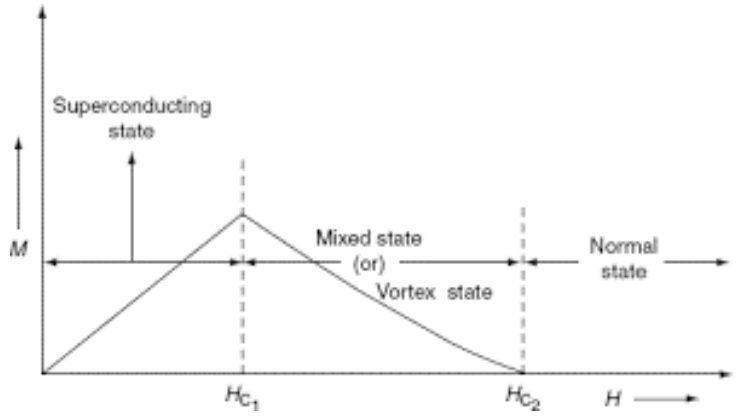


DEPENDENCE OF MAGNETIC MOMENT ON *H*

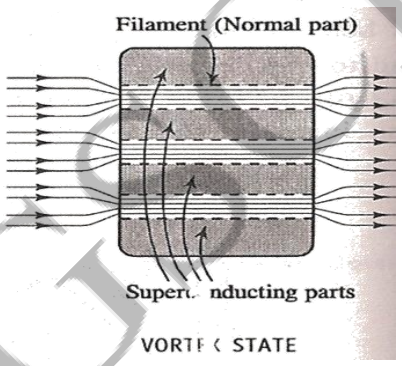
Type-II superconductors

Type-II superconductors are hard superconductors. They exist in three states

1) Superconducting state
2) Mixed state
3) Normal state

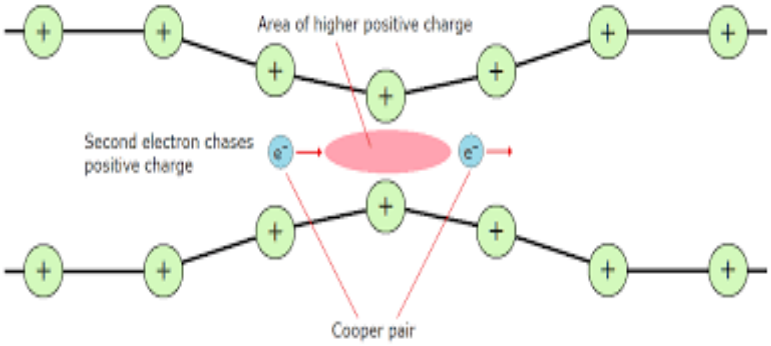


They are having two critical fields H_{c1} and H_{c2}. For the field less then H_{c1}, it expels the magnetic field completely and becomes a perfect diamagnetic. Between H_{c1} and H_{c2} the flux starts penetrating throughout the specimen. This state is called vortex state. H_{c2} is 100 times higher than Hc1. At H_{c2} the flux penetrates completely and becomes normal conductor. Type-II superconductors are used in the manufacturing of the superconducting magnets of high magnetic fields.



4.2.5. BCS Theory:

Bardeen, Cooper and Schrieffer (BCS) in 1957 explained the phenomenon of superconductivity based on the formation of cooper pairs. It is called BCS theory. It is a quantum mechanical concept.



When a current flow in a superconductor, electrons come near a positive ion core of lattice, due to attractive force. The ion core also gets displaced from its position, which is called *lattice distortion*. The lattice vibrations are quantized in a term called *Phonons*. Now an electron which comes near that place will interact with the distorted lattice. This tends to reduce the energy of the electron. It is equivalent to interaction between the two electrons through the lattice. This leads to the formation of cooper pairs.

"*Cooper pairs* are a bound pair of electrons formed by the interaction between the electrons with opposite spin and momenta in a phonon field". According to quantum mechanics a cooper pair is treated as single entity. A wave function is associated with each cooper pair. This holds good over a large volume with finite value for its amplitude. The wave function of similar cooper pairs overlaps. For one cooper pair overlapping may extend over 10^6 other pairs. Thus, it covers entire volume of the superconductor. It leads to union of large number of cooper pairs. The resistance encountered by any single cooper pair is overcome by combined action of other pairs in the union.

When the electrons flow in the form of cooper pairs in materials, they do not encounter any scattering and the resistance factor vanishes or in other words, conductivity becomes infinity, which is called as superconductivity.

In superconducting state electron-phonon interaction is stronger than the coulomb force of attraction of electrons. Cooper pairs are not scattered by the lattice points. They travel freely without slow down as their energy is not transferred. Due to this, they do not possess any electrical resistivity.

4.2.6. Effect of magnetic field:

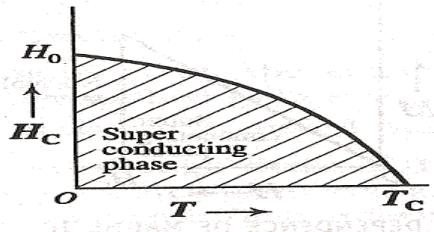
Superconductivity can be destroyed by applying magnetic field. The strength of the magnetic field required to destroy the superconductivity below the T_c is called critical field. It is denoted by $H_c(T)$.

If 'T' is the temperature of the superconducting material, 'T_c' is the critical temperature, 'H_c' is the critical field and 'H_o' is the critical field at 0^o K.

They are related by

$$H_c = H_o\left[1-\left(\frac{T}{T_c}\right)^2\right]$$

By applying magnetic field greater than H_o, the material can never become superconductor whatever may be the low temperature. The critical field need not be external but large current flowing in superconducting ring produce critical field and destroys superconductivity.



DEPENDENCE OF H_c ON T

High temperature superconductors:

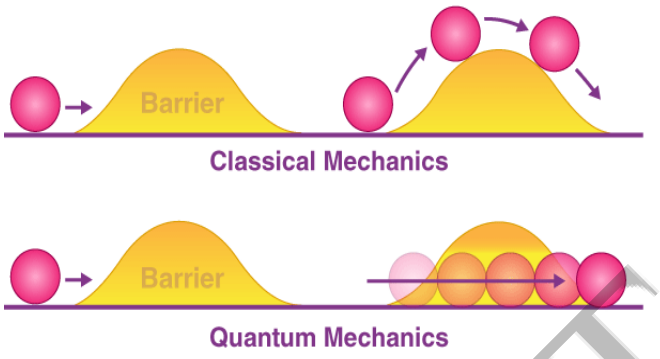
The term high-temperature superconductor was first used to designate the new family of cuprite-perovskite ceramic materials discovered by Bednorz and Müller in 1986. The first high-temperature superconductor, LaBaCuO, with a transition temperature of 30 K and in the same year LSCO ($La_{2-x}Sr_xCuO_4$) discovered with T_C of 40K. In 1987 it was shown that superconductors with T_c greater than 77K could be prepared, this temperature is greater than the liquid helium temperature. $YBa_2Cu_3O_7$ was discovered to have a Tc of 92 K. Bismuth/lead strontium Calcium Copper $(Bi\ Pb)_2Sr_2\ Ca_2Cu_3O_x$ ($x<0.1$) with T_c=105K. Thallium barium Calcium copper oxide ($Tl\ Ba_2Ca_2Cu_3O_4$) of T_c=115K. Mercury barium calcium copper oxide ($Hg\ Ba_2Ca_2Cu_3O_4$) with T_c=135K.

All high temperature superconductors are different types of oxides of copper, and bear a particular type of crystal structure called Perovskite crystal structure. The number of copper layers increases the T_c value increases. The current in the high T_C materials is direction dependent. It is strong in parallel to copper-oxygen planes and weak in perpendicular to copper-oxygen planes.

High T_c materials are Type-II superconductors and they are brittle and don't carry enough current. The formation of electron pairs is not due to interaction of electron lattice as in the BCS theory. Still it is not clear what does cause the formation of pairs. Research is being conducting in this direction. The high temperature superconductors are useful in

high field applications. It can carry high currents of 10^5 to 10^6 amps in moderate magnetic fields. They are used in military applications, Josephson junction in SQUIDS, under sea communication, submarines.

4.2.7. Quantum Tunneling:



What is Quantum Tunnelling?

- Quantum tunnelling is defined as a quantum mechanical process where wavefunctions can penetrate through a potential barrier.
- The transmission through the potential barrier can be finite and relies exponentially on the barrier width and barrier height.
- No wave or particle is eliminated. Tunnelling happens with barriers of thickness about 1–3 nm and smaller. Quantum tunnelling cannot be explained through the laws of classical mechanics, where a dense potential barrier needs potential energy.
- It has a crucial role in physical processes such as nuclear fusion. It's been used in quantum computing, tunnel diodes and scanning tunnelling microscopes.
- In other words, the uncertainty in the precise location of electromagnetic particles permits these particles to break the laws of classical physics and propagate in space without going over the potential energy boundary.
- Both tunnelling and uncertainty principle are mutually compatible as they consider a quantum body as both wave and particle simultaneously.

4.2.8. Josephson Effect

Josephson effect is perhaps the occurrence which happens when two superconductors are located in proximity and have some hindrance between them. Moreover, the Josephson effect builds a current which is called super current. This super current flows continuously without any voltage across a device that is called Josephson junction.

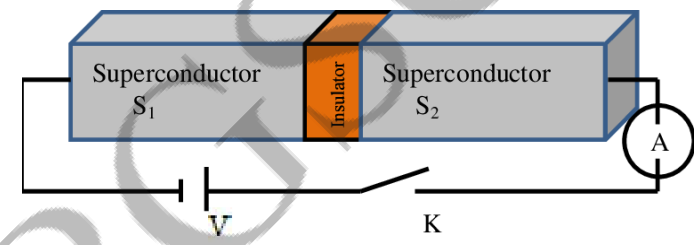
The Josephson effect is a quantum mechanical phenomenon that occurs in superconductors, which are materials that conduct electricity with zero resistance at very low temperatures. There are two types of Josephson effects: the AC Josephson effect and the DC Josephson effect.

4.2.9. The DC Josephson Effect

As the name suggests, the DC Josephson effect is related to the direct current crossing, which goes over the insulator when there is no external electromagnetic field present. DC Josephson effect refers to the phenomenon of a supercurrent flowing between two superconductors that are separated by a thin insulating barrier, without any applied voltage. In this effect, the supercurrent flows steadily, or "direct current," through the Josephson junction.

The AC Josephson Effect

The AC Josephson effect refers to the phenomenon of a supercurrent flowing between two superconductors that are separated by a thin insulating barrier, also called a Josephson junction, when a voltage oscillating at a particular frequency is applied to the junction. This effect is known as AC because the voltage oscillates with time, producing an alternating current in the junction.



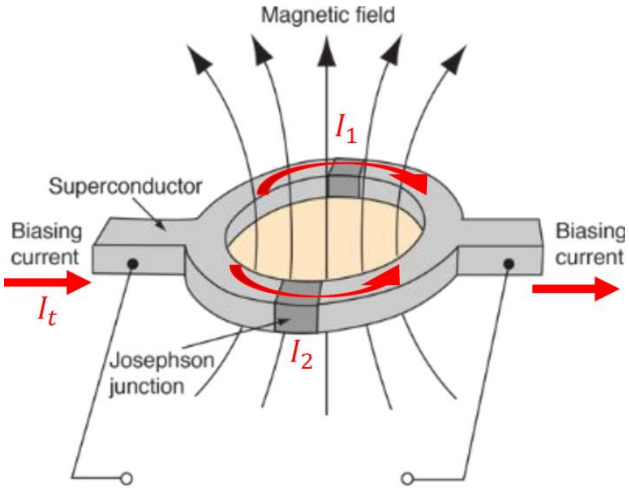
4.2.10. SQUID (Superconducting Quantum Interference Device)

A SQUID (superconducting quantum interference device) is a very sensitive magnetometer which works through the Josephson effect

(We know that a small charge in magnetic field produces variation in the flux quantum.)

It is an Ultra –sensitive instrument used to measure very weak magnetic field of the order of the 10^{-14} tesla.

- It consists of a superconducting ring which can have the magnetic field of quantum values (1,2, 3,.) of flux placed in between two Josephson junctions as shown in figure.



Superconducting Ring

- When the magnetic field is applied perpendicular to the plane of the ring, the current is induced at the two Josephon junctions.
- The induced current current produces the interference pattern and if flows around the ring so that the magnetic flux in the ring can have the quantum value of magnetic field applied.

Applications:

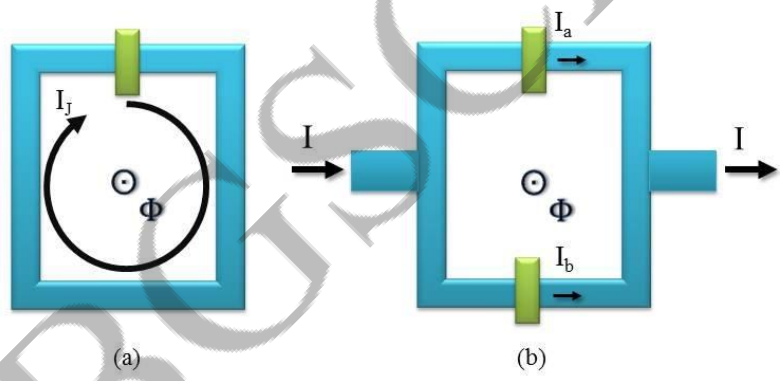
- SQUID can be used to defect the variation of very minute magnetic signals in terms of quantum flux.
- It is used as a storage device for magnetic flux.
- It is used to study earth quakes and to remove paramagnetic impurities

DC SQUID

- Diagram of a DC SQUID. The current enters and splits into the two paths, each with currents $I_a$and I_b.
- The thin barriers on each path are Josephson junctions, which together separate the two superconducting regions. Φ represents the magnetic flux threading the DC SQUID loop.
- It has two Josephson junctions in parallel in a superconducting loop. It is based on the DC Josephson effect. In the absence of any external magnetic field, the input current I splits into the two branches equally. If a small external magnetic field is applied to the superconducting loop, a screening current I_s begin to circulate the loop that generates the magnetic field cancelling the applied external flux, and creates an additional Josephson phase which is proportional to this external magnetic flux.

RF SQUID

- A radio frequency (RF) SQUID is made up of one Josephson junction, which is mounted on a superconducting ring. An oscillating current is applied to an external circuit, whose voltage changes as an effect of the interaction between it and the ring. The magnetic flux is then measured.
- It is based on the AC Josephson effect and uses only one Josephson junction. It is less sensitive compared to DC SQUID but is cheaper and easier to manufacture in smaller quantities. Most fundamental measurements in bio magnetism, even of extremely small signals, have been made using RF SQUIDS. The RF SQUID is inductively coupled to a resonant tank circuit.
- Depending on the external magnetic field, as the SQUID operates in the resistive mode, the effective inductance of the tank circuit changes, thus changing the resonant frequency of the tank circuit. These frequency measurements can be easily taken, and thus the losses which appear as the voltage across the load resistor in the circuit are a periodic function of the applied magnetic flux with a period of Φ_0



RF SQUID & DC SQUID

4.2.11. Application in quantum computing: charge, Phase and Flux qubit:

Each type of qubit has its own advantages and disadvantages, and they are used in different applications.

Charge qubits use the charge of an electron as the basis for quantum information processing. These qubits are simple to fabricate and operate, and they have relatively long coherence times. Charge qubits are useful for performing simple quantum algorithms, but they are not as scalable as other types of qubits.

Flux qubits use the magnetic flux through a superconducting loop as the basis for quantum information processing. These qubits have the advantage of being highly tunable, which allows for precise control of their behavior. Flux qubits are also relatively robust against environmental noise. However, they have a short coherence time compared to other qubit types.

Phase qubits use the phase difference between two superconducting currents as the basis for quantum information processing. These qubits are simple to operate and have relatively long coherence times. Phase qubits are useful for performing simple quantum algorithms, and they are also scalable to larger systems.

Overall, each type of qubit has its own strengths and weaknesses, and researchers choose the type of qubit that is best suited for the application they are working on. Quantum computing is still a relatively new field, and there is ongoing research to develop new types of qubits with even better performance characteristics.

Solved Problems:

1. Find the temperature at which there is 1.0% probability that a state with an energy 0.5 eV above Fermi energy will be occupied.

Solution:

The Fermi distribution function is

$$f(E)= \frac{1}{1+e^{\frac{E-E_f}{KT}}}$$

f(E) = 1.0% = 0.01

$E-E_F$ = 0.5 eV = $0.5\times1.6\times10^{-19}$ J

$E-E_F = 8\times10^{-20}$ J

k = 1.38×10-23 J/K

$$0.01 = \frac{1}{1+e^{\frac{8\times10^{-20}}{1.38\times10^{-23}T}}}$$

$$1+e^{\frac{5797.1}{T}} = 100$$

$$e^{\frac{5797.1}{T}} = 99$$

$$\frac{5797.1}{T} = 4.595$$

T = 1261.6 K

2. Calculate the probability of an electron occupying an energy level 0.02 eV above the Fermi level at 200K and 400K in a material.

Solution:

The fermi distribution function is

$$f(E) = \frac{1}{1+e^{\frac{E-E_f}{KT}}}$$

$E-E_F = 0.02$ eV $= 0.02 \times 1.6\times10^{-19}$ J $= 3.2\times10^{-21}$

$k = 1.38\times10^{-23}$ J/K

For T = 200K,

$$f(E) = \frac{1}{1+e^{\frac{3.2\times10^{-21}}{1.38\times10^{-23}x200}}}$$

f(E) = 0.2388

For T = 400K

$$f(E) = \frac{1}{1+e^{\frac{3.2\times10^{-21}}{1.38\times10^{-23}x400}}}$$

f(E) = 0.359

3. Calculate the probability of an electron occupying an energy level 0.02 eV above the fermi level and 0.02 eV below the fermi level at 200K.

Solution:

For 0.02 eV below the Fermi level at 200K

$E-EF = -0.02$ eV $= -0.02 \times 1.6\times10^{-19}$ J $= -3.2\times10^{-21}$

k = 1.38×10-23 J/K

T = 200K

$$f(E) = \frac{1}{1+e^{\frac{E-E_f}{KT}}}$$

$$f(E) = \frac{1}{1+e^{\frac{-3.2\times10^{-21}}{1.38\times10^{-23}x200}}}$$

f(E) = -0.761

For 0.02 eV above the Fermi level at 200K

$$f(E) = \frac{1}{1+e^{\frac{3.2\times10^{-21}}{1.38\times10^{-23}x200}}}$$

f(E) = 0.2388

4. Superconducting tin (Sn) has a critical temperature of 3.7 K at Zero magnetic field and a critical field of 0.0306 tesla at 0K. Find critical field at 2K. (Model QP1-2023)

Solution:

Critical temperature Tc = 3.7 K

Critical magnetic field at 0K, H0 = 0.0306 tesla

Temperature T=2K

Critical field at T Hc =?

$H_c = H_o\left[1-\left(\frac{T}{T_c}\right)^2\right]$

$H_c = 0.0306\left[1-\left(\frac{2}{3.7^2}\right)^2\right]$

$H_c = 0.0306\left[1-\left(\frac{4}{13.69}\right)\right]$

$H_c = 0.0306\left[13.69 - \left(\frac{4}{13.69}\right)\right]$

H_c = 0.0217 tesla

5. Lead has a super conducting transition temperature of 7.26 K. If initial field at 0K is $50\times10^3 Am^{-1}$, calculate the critical field at 6K. (Model QP1-2023)

Solution:

Transition temperature, ***Tc***= 7.26 K & ***T*** = 6 K

Field at 0K, i.e., $\mathbf{H_0}$= 50 X 10^3 Am−1

To find: $\boldsymbol{CriticalFeild\ Hc}$ =?

We have, $H_c = H_o\left[1-\left(\frac{T}{T_c}\right)^2\right]$

$\boldsymbol{H_c = 50\times10^3\left[1-\left(\frac{6}{7.26}\right)^2\right] = 50\times10^3\,[1-0.623]}$

$\boldsymbol{Hc = 18.834\times10^3}$Am^{-1}

6. The super conducting transition temperature of Lead is 7.26 K initial field at 0 K is 64 X 10^3 Am^{-1}, calculate the critical field at 5K.

Solution:

Transition temperature, $\boldsymbol{Tc}$= 7.26 K & T = 5 K

Field at 0K, i.e., $\mathbf{H_0}$= 64 X 10^3 Am−1

To find: $\boldsymbol{CriticalFeild\ Hc}$ =?

We have, $H_c = H_o\left[1-\left(\frac{T}{T_c}\right)^2\right]$

$\boldsymbol{H_c}=64\times\boldsymbol{10^3}\left[1-\left(\frac{5}{7.26}\right)^2\right] = 64\times\boldsymbol{10}\ [\boldsymbol{1}-\boldsymbol{0}.474]$

$\boldsymbol{Hc}$=33.66×$\boldsymbol{10^3}$Am^{-1}

7. A super conducting material has a critical temperature of 3.7 K in zero magnetic field and a critical field of 0.02 T at 0K. Find the critical field at 3K.

Solution:

Transition temperature, **Tc** = 3.7 K & T = 3 K

Field at 0K, i.e., H_0= 0.02 Tesla Am−1

To find: $\boldsymbol{CriticalFeild\ Hc}$ =?

We have, $H_c = H_o\left[1-\left(\frac{T}{T_c}\right)^2\right]$

$\boldsymbol{H_c}=0.02\left[1-\left(\frac{3}{3.7}\right)^2\right] = 0.02\ [\boldsymbol{1}-\boldsymbol{0.6}57]$

$\boldsymbol{Hc}$ =0.013 × $\boldsymbol{10^3}$ Tesla

8. The critical field for lead is 1.2×10^5 A/m at 8K and 2.4×10^5 A/m at 0K. Find the critical temperature of the material.

Solution:

Critical field**, Hc** = **1.2 ×10^{5}A/m & T =** 8 K

Field at 0K, i.e., **H_0** = **2.4 ×10^5 A/m**

$$H_c = H_o\left[1-\left(\frac{T}{T_c}\right)^2\right]$$

$$\mathbf{1.2\times10^5} = \mathbf{2.4\times10^5}\left[1-\left(\frac{8}{Tc}\right)^2\right] = 0.02\ [\mathbf{1-0.6}57]$$

$$\frac{1.2\times10^5}{2.4\times10^5} = \left[1-\left(\frac{8}{Tc}\right)^2\right]$$

$$\frac{64}{Tc^2} = \mathbf{1-0.5}$$

$$\boldsymbol{T}^2\mathrm{C} = \frac{64}{0.5}$$

$$\boldsymbol{T_c} = \mathbf{11.3}\mathrm{K}$$

9. The critical field for lead are 1.2×10^{5}A/m & 3.6×10^5 A/m at 12K & 10K, respectively. Find its critical temperature & critical field at 0K & 3.2K.

Data: Critical field, Hc(T_1) = 1.2 ×105A/m, T_1= 12 K

& Hc(T_2) = 3.6 ×10^{5}A/m, T_2= 10 K

To find: $\boldsymbol{CriticalTemperature\ Tc}$=? & $\boldsymbol{f}$ield $\boldsymbol{H_{0K}}$=? & $\boldsymbol{H_{3.2K}}$=?

$$H_c = H_o\left[1-\left(\frac{T}{T_c}\right)^2\right]$$

$$1.2\times10^5 = H_o\left[1-\left(\frac{12}{T_c}\right)^2\right]\text{----------(1)}$$

$$H_c\ (\boldsymbol{T_1}) = H_o\left[1-\left(\frac{T_1}{T_c}\right)^2\right]$$

$3.6\times10^{5} = H_o\left[1-\left(\frac{10}{T_c}\right)^2\right]$----------(2)

On dividing Eqn. (1) by (2),

$$\frac{1}{3} = \frac{1-\left(\frac{12}{T_c}\right)^2}{1-\left(\frac{10}{T_c}\right)^2} \rightarrow 1-\left(\frac{10}{T_c}\right)^2 = 3-3\left(\frac{12}{T_c}\right)^2 \rightarrow \frac{432-100}{T_c^2} \rightarrow \boldsymbol{Tc=12.88K}$$

Put in Eqn. (1), $1.2\times10^{5} = H_o\left[1-\left(\frac{12}{12.88}\right)^2\right] \rightarrow H_o = \frac{1.2\times10^{5}}{1-\left(\frac{12}{12.88}\right)^2}$ $\boldsymbol{H0 = 9.06\times10^{5} A/m}$

Now for T=3.2K,

$H_c = H_o\left[1-\left(\frac{T}{T_c}\right)^2\right] = \mathbf{9.06\times10^{5}}\left[1-\left(\frac{3.2}{12.88}\right)^2\right] = H_{3.2} = 8.5\times\mathbf{10^{5}}\boldsymbol{A/m}$

MODULE -5

APPLICATIONS OF PHYSICS IN COMPUTING

5.1. Physics of Animation

Animation

Animation is a method of photographing successive drawings, models, or even puppets, to create an illusion of movement in a sequence. Because our eyes can only retain an image for approx. $1/10^{th}$ of a second, when multiple images appear in fast succession, the brain blends them into a single moving image. Animation is the process of displaying still images in a rapid sequence to create the illusion of movement.

5.1.1. The Taxonomy of Physics-Based Animation Methods

At the highest level, the field of physics-based animation and simulation can roughly be subdivided into two large groups:

1. Kinematics is the study of motion without consideration of mass or forces.
2. Dynamics is the study of motion taking mass and forces into consideration.

kinematics and dynamics come in two flavours or subgroups:

1. Inverse is the study of motion knowing the starting and ending points.
2. Forward is the study of motion solely given the starting point.

5.1.2. Frames

A frame is a single image in a sequence of pictures. A frame contains the image to be displayed at a unique time in the animation. In general, one second of a video is comprised of 24 or 30 frames per second also known as FPS. The frame is a combination of the image and the time of the image when exposed to the view. An extract of frames in a row makes the animation.

5.1.3. Frames per Second

Animation shot on film and projected is played at 24 frames per second. Animation for television in Europe, Africa, the Middle East and Australia is played at 25 frames per second.

Sl. No.	**System**	**Frames Per Second**
1	PAL (Australia, Middle East, Africa)	25
2	NTSC(America, West Indies, Specific Rim Countries)	30

An animated film with 25 frames per second is played on television at 24 frames per second would result in a black bar rolling up the screen. Then Digital Converts are to be used to

transfer one speed of film to another speed of video. The most important thing to find out when animating something is what speed the animation will be played back at.

5.1.4. Size and Scale

The size and scale of characters often play a central role in a story's plot. What would Superman be without his height and bulging biceps? Some characters, like the Incredible Hulk, are even named after their body types.

We often equate large characters with weight and strength, and smaller characters with agility and speed. There is a reason for this. In real life, larger people and animals do have a larger capacity for strength, while smaller critters can move and maneuver faster than their large counterparts. When designing characters, you can run into different situations having to do with size and scale, such as:

1. Human or animal-based characters that are much larger than we see in our everyday experience. Superheroes, Greek gods, monsters,
2. Human or animal-based characters that are much smaller than we are accustomed to, such as fairies and elves.
3. Characters that need to be noticeably larger, smaller, older, heavier, lighter, or more energetic than other characters.
4. Characters that are child versions of older characters. An example would be an animation featuring a mother cat and her kittens. If the kittens are created and animated with the same proportions and timing as the mother cat, they won't look like kittens; they'll just look like very small adult cats.

Proportion and Scale

Creating a larger or smaller character is not just a matter of scaling everything about the character uniformly. To understand this, let's look at a simple cube. When you scale a cube, its volume changes much more dramatically than its surface area. Let us say each edge of the cube is 1 unit length. The area of one side of the cube is 1 square unit, and the volume of the cube is 1 cubed unit. If you double the size of the cube along each dimension, its height increases by 2 times, the surface area increases by 4 times, and its volume increases by 8 times. While the area increases by squares as you scale the object, the volume changes by cubes.

5.1.5. Wight and strength

Body weight is proportional to volume. The abilities of your muscles and bones, however, increase by area because their abilities depend more on cross-sectional area than volume. To increase a muscle or bone's strength, you need to increase its cross- sectional area. To double a muscle's strength, for example, you would multiply its width by √2. To triple the strength, multiply the width by √3. Since strength increases by squares and weight increases by cubes, the proportion of a character's weight that it can lift does not scale proportionally to its size.

Let us look at an example of a somewhat average human man. At 6 feet tall, he weighs 180 pounds and can lift 90 pounds. In other words, he can lift half his body weight. If you scale up the body size by a factor of 2, the weight increases by a factor of 8. Such a character could then lift more weight. But since he weighs more than 8 times more than he did before, he can not lift his arms and legs as easily as a normal man. Such a giant gains strength, but loses agility.

5.1.6. Motion and Timing in animations

Introduction to Motion:

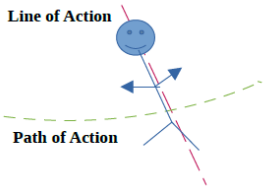
Motion is an essential component in games and animations. The motion is governed by the newtons laws and kinematic equations. When animating a scene, there are several types of motion to consider. These are the most common types of motion:

1. Linear
2. Parabolic
3. Circular
4. Wave

Motion and timing go hand in hand in animation.

Motion Lines and Paths

Individual drawings or poses have a line of action, which indicates the visual flow of action at that single image. Motion has a path of action, which indicates the path along which the object or character moves. The path of action refers to the object's motion in space. While it can help show timing, its primary function is to see the direction and path of the motion, and not necessarily its timing.



Timing

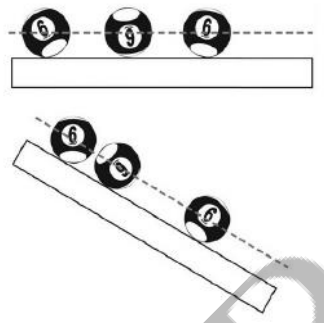
The timing is the choice of when something should be done; the regulation of occurrence and pace to achieve a desired effect. Animators have the ability to move forward and backward in time to place objects when and where they are to be.

Timing Tools

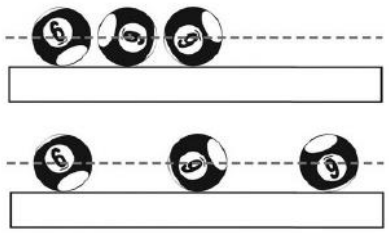
In animation, timing of action consists of placing objects or characters in particular locations at specific frames to give the illusion of motion. Animators work with very small intervals of time; most motion sequences can be measured in seconds or fractions of seconds. Frame intervals between keys are usually smaller than one second.

Linear Motion Timing

Linear motion refers to motion in a straight line, always in the same direction. An object moving with linear motion might speed up or slow down as it follows a linear path. A heavy ball rolling on a table or incline is an example of linear motion. The all is rotating, but its center of gravity follows a linear path. A heavy ball rolling on a table or incline is an example of linear motion. The ball is rotating, but its center of gravity follows a linear path.



Linear motion



Uniform motion

Uniform Motion Timing

When uniform motion occurs, the net force on the object is zero. Net force is the total of all forces added up. There might be several forces acting on the object, but when both the magnitude and direction of the forces are added up, they add up to zero. Uniform motion is the easiest to animate because the distance the object travels between frames is always the same. Uniform motion is a type of linear motion with constant speed and no acceleration or deceleration. The object moves the same distance between consecutive frames. The longer the distance between frames, the higher the speed.

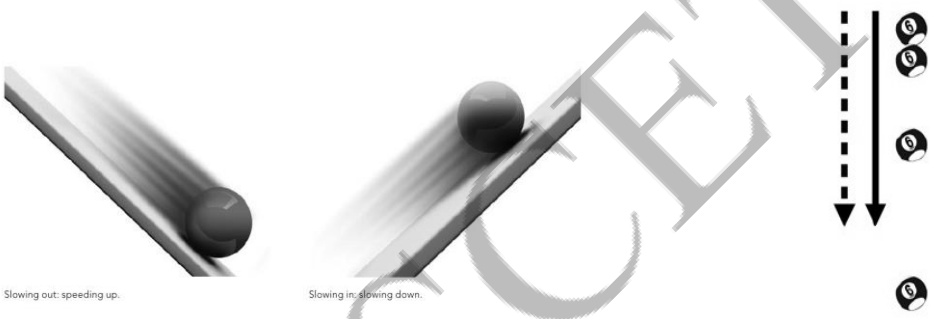
Slow in and Slow out

When motion is accelerating or decelerating, we refer to this type of motion as a slow in or slow out. This type of motion is sometimes called ease in or ease out. In this book, we use the hyphenated forms slow-in and slow-out for easier understanding.

1. Slow in, ease in—The object is slowing down, often in preparation for stopping.
2. Slow out, ease out—The object is speeding up, often from a still position.

The term slow out can be confusing, since it essentially means "speed up." one can think of slow out as the same as ease out, as in easing out of a still position and speeding up to full speed.

For example, a ball rolling down an incline or dropping straight down is slowing out, as it goes from a still position or slow speed to a fast speed. A ball rolling up an incline is slowing in.



Acceleration Timing

Timing for acceleration can be calculated very accurately when the net force being exerted is constant. Let's take a look at the forces and how they can be used to calculate the animation's timing.

5.1.7. Constant Forces

A constant force is a force that doesn't vary over time. Examples of constant forces include:

1. Gravity pulling an object to the ground
2. Friction bringing an object to a stop

Constant force and Acceleration

Constant forces result in constant acceleration. Because the acceleration is constant, we can figure out the timing for such sequences using a few principles of physics.

The resulting acceleration depends on the direction of the force and motion, if there is any motion at all to begin with.

1. When constant net force is applied to an unmoving object, the result is acceleration.

2. When constant net force is applied to a moving object in the same direction as the motion, the result is acceleration.
3. When constant net force is applied in the direction opposite the existing motion, the result is deceleration (acceleration in the opposite direction).

Forces Exerted by Characters

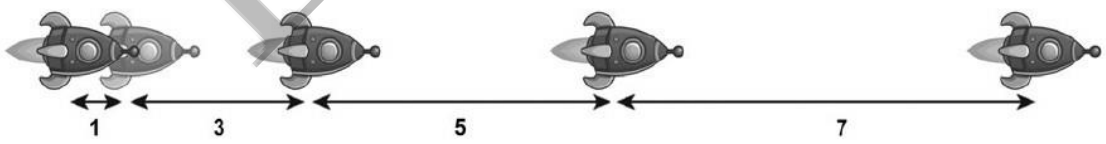
Forces exerted by people's bodies are rarely constant throughout an entire motion. For the purposes of animation, however, one can break the character motion into short time segments and consider each of these segments to be responding to constant net force. This will make it easier for one to calculate the timing for each individual segment.

As an example, let's look at the push for a jump. The force a character exerts during the push is somewhat constant, and the timing is very short (less than half a second). In such a case the timing for a constant force is an excellent starting point, and in most cases will do the job as is.

A character walking and pushing a rock is not exerting a constant force throughout the entire sequence, but during each short part of the walk cycle the net force could be considered to be a different constant value.

5.1.8. The Odd Rule

When acceleration is constant, one can use the Odd Rule to time the frames. With this method, one calculates the distance the object moves between frames using a simple pattern of odd numbers. Between consecutive frames, the distance the object moves are a multiple of an odd number. For acceleration, the distance between frames increases by multiples of 1, 3, 5, 7, etc.



Rocket speeding up using the Odd Rule.

For deceleration, the multiples start at a higher odd number and decrease, for example 7, 5, 3, 1.

Sled coming to a stop using the Odd Rule.

The Odd Rule is a multiplying system based on the smallest distance traveled between two frames in the sequence. For a slow-out, this is the distance between the first two frames; for a slow-in, it's the distance between the last two frames.

This distance, the base distance, is used in all Odd Rule calculations.

Odd Rule Multipliers

The Odd Rule in its simplest form, as described above, is just one way to use it. For example, one can instead calculate the distance from the first frame to the current frame and use these distances to place the object on specific frames.

	Multiply by base distance to get distance between:	
Frame #	Consecutive frames	First frame and this frame
1	n/a	0
2	1	1
3	3	4
4	5	9
5	7	16
6	9	25
7	11	36

calculating the distance for a large number of frames and a chart like this isn't practical, one can figure out the oddbnumber multiplier for consecutive frames with this formula:

Odd number multiplier for consecutive frames = ((frame # – 1) * 2) – 1

In the charts above, note that the distances in the last column are squared numbers: 4 = 22, 9 = 32, 16 = 42, and so on. One of the benefits of the Odd Rule is one can calculate the total distance travelled from the start point to the current frame with the following formula:

Multiplier for distance from first frame to current frame = (current frame # – 1) 2

When setting the keys, one can use either the consecutive key multipliers or total distance multipliers but need to Choose the one that's easiest to use for the animated sequence.

5.1.9. Odd Rule Scenarios

Here are a few different scenarios for calculating the distance an object travels between keys in a slow-in or slow-out.

Base Distance Known Speeding up

If the object is speeding up, the first frame distance is the base distance. If one knows the base distance, figuring out the distance the object travels at each frame is pretty

straightforward. Just multiply the base distance by 3, 5, 7, etc. to get the distances between consecutive frames, or use squares to multiply the base distance to get the total distance travelled on each frame.

Base Distance Known Slowing Down

Suppose one wants an object to slow down, and one knows the distance between the last two frames before it stops. For slow-ins, the base distance is the distance between the last two frames. The solution is to work backward, as if the object were speeding up in the opposite direction. Working backward, multiply the base distance by 3, 5, 7, etc. to get the distances between each previous frame in the sequence.

Total Distance and Number of Frames Known, Speeding Up

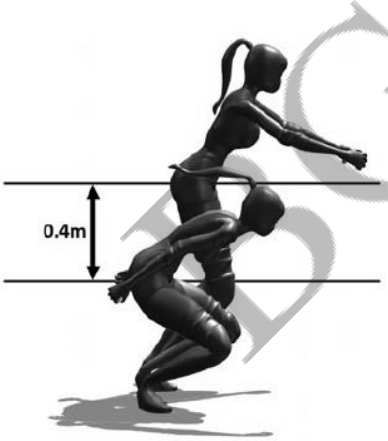
If one wants know the total distance and the total number of frames, one can find the base distance with this formula:

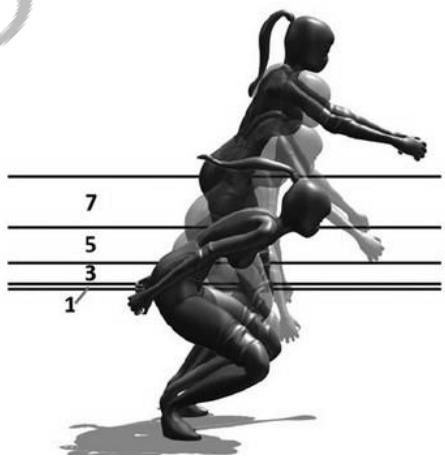
Base distance = Total distance/(Last frame number – 1) 2

Suppose there is a jump push (takeoff) with constant acceleration over 5 frames, and the total distance traveled is 0.4m.

Using the formula above, we find the base distance.

Base distance = 0.4m/(5 – 1) 2 = 0.4m/16 = 0.025m



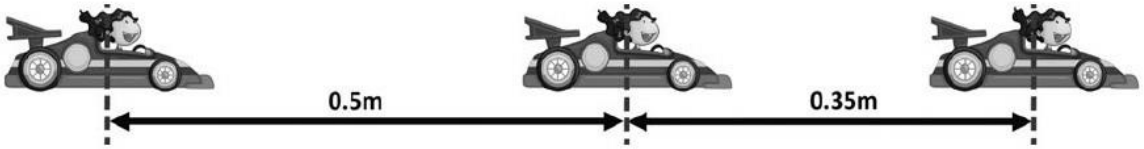


Using the base distance, one can calculate the distances between each frame. If one adds up the distances travelled, one will find that they add up to exactly 0.4m.

First Key Distance Known Slowing Down

Suppose one has a moving object that one wants to slow down, and one has set the first frame of the slow-in to give an idea of the pacing for the sequence. In this case, one can consider that the distance the object moved between the last two frames before the slow-in is part of

the calculation— the distance between them becomes the first frame distance, and the first slow-in frame becomes the second frame in the sequence.



One feature of the Odd Rule is that the base distance is always half the difference between any two adjacent distances.

To find the base distance, one can simply calculate:

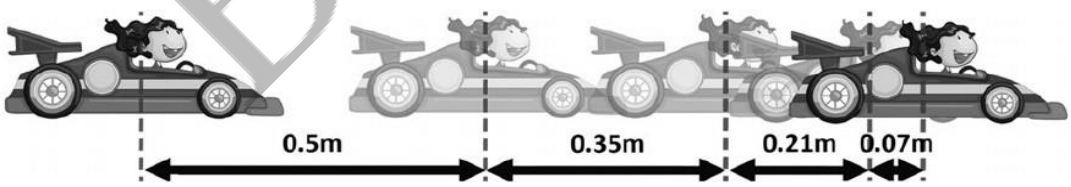
$$(0.5\text{m} - 0.35\text{m})/2 = 0.07\text{m}$$

To figure out how many frames are in the slow-in, divide the first distance by the base distance to find out which odd number it corresponds to.

$$0.5/0.07 = 7$$

This means the first distance corresponds to 7 in the 7, 5, 3, 1 sequence, making the sequence four frames long. Now ibe can work back the other way, multiplying the base distance by odd numbers to get the distances for the rest of the slow in frames.

Frame #	Consecutive frame multiplier	Distance from previous frame
1	7	7 * 0.07m = 0.5m
2	5	5 * 0.07m = 0.35m
3	3	3 * 0.07m = 0.21m
4	1	1 * 0.07m = 0.07m



5.1.10. Motion Graphs

A motion graph plots an object's position against time. If one is using animation software, understanding and using motion graphs is a key skill in animating anything beyond the simplest of motions. If one is drawing the animation, drawing motion graphs before animating can help one to visualize the motion. On a motion graph, the time goes from left to right across the bottom of the graph, while the object's position is plotted vertically against

the time. Each axis in 3D space (X, Y, Z) has its own line showing the object's position along that axis. At the very least, one will need to understand the types of lines in a motion graph and what they represent in terms of visible motion. one can also look at motion graphs to get a better understanding of any difficulties one is having with the timing or action.

5.1.11. Examples of Character Animation

5.1.11.1. Jumping

A jump is an action where the character's entire body is in the air, and both the character's feet leave the ground at roughly the same time. A jump action includes a takeoff, free movement through the air, and a landing.

5.1.11.2. Parts of Jump

A jump can be divided into several distinct parts:

- **Crouch**—A squatting pose taken as preparation for jumping.
- **Take off**—Character pushes up fast and straightens legs with feet still on the ground. The distance from the character's center of gravity (CG) in the crouch to the CG when the character's feet are just about to leave the ground is called the push height. The amount of time (or number of frames) needed for the push is called the push time.
- **In the air**—Both the character's feet are off the ground, and the character's center of gravity (CG) moves in a

parabolic arc as any free-falling body would. First it reaches an apex, and then falls back to the ground at the same rate at which it rose. The height to which the character jumps, called the jump height, is measured from the CG at takeoff to the CG at the apex of the jump. The amount of time the character is in the air from take off to apex is called the jump time. If the takeoff pose and the landing pose are similar, then the jump height and jump time are about the same going up as they are going down.

- **Landing**—Character touches the ground and bends knees to return to a crouch. The distance from the character's CG when her feet hit to the ground to the point where the character stops crouching is called the stop height. The stop height is not always exactly the same as the push height.

Calculating Jump Actions

When working out the timing for a jump, one will need to first decide on:

1. Jump height or jump time
2. Push height

3. Stop height

4. Horizontal distance the character will travel during the jump

From these factors, one can calculate the timing for the jump sequence.

Calculating Jump Timing

When planning the jump animation, the most likely scenario is that you know the jump height, expressed in the units you are using for the animation (e.g., inch or cm).

Placement and timing for frames while the character is in the air follow the same rules as any object thrown into the air against gravity. Using the tables in the Gravity chapter (or an online calculator), one can figure out the jump time for each frame. Look up the amount of time it takes an object to fall that distance due to gravity, and express the jump time in frames based on the fps one is using.

Example:

Jump height = 1.2 m

Jump time for 1.2 m = 0. 5 seconds

Jump time at 30fps = 0. 5 * 30 = 15 frames

5.1.11.3. Jump Magnification

When calculating the remainder of the timing for the entire jump action, you can use a factor called jump magnification (JM). The JM can be used to calculate the push timing and stop timing.

The JM is the ratio of the jump height to the push height.

$$JM = \frac{JumpHeight}{PushHeight}$$

Since you already know the jump height and push height, you can calculate the JM. Then you can use the JM to calculate other aspects of the jump.

Example:

Jump Height = 1m

Push Height = 0.33m

JM = Jump Height/Push Height = 3

Jump Magnification and Acceleration

Jump Magnification is in fact an exact ratio that tells one how much the character has to accelerate against gravity to get into the air. The JM, besides being the ratio of jump-to-push vertical height and time, is also the ratio of push-to-jump vertical acceleration. Opposite the

other ratios: while a longer jump time means a shorter push time, a higher jump acceleration means a much, much higher push acceleration. Knowing about this can help you make more informed decisions about your push timing.

To see how this works, let's look at the formula for JM and relate it to acceleration:

Jump Time Jump Height

$$JM = \frac{Jump\ Time}{Push\ Time} = \frac{Jump\ Height}{Push\ Height} = \frac{Push\ Acceleration}{Jump\ Acceleration}$$

The magnitude of jump acceleration is always equal to gravitational acceleration, with deceleration as the character rises and acceleration as it falls.

$$JH = \frac{Push\ Acceleration}{Jump\ Acceleration} = \frac{Push\ Acceleration}{Gravitational\ Acceleeration}$$

Your landing speed is the same as the velocity of any falling object, which you can easily calculate from the free fall time. Since acceleration due to gravity is 10m/sec^2, this means that after one second a falling object is traveling at 10m/sec, after two seconds at 20m/sec, after three seconds at 30m/sec, and so on. Since takeoff speed is the same as landing speed, you need to get up to that same speed when taking off for a jump. If your landing speed is 10m/sec, then during your takeoff you need to get up to a speed of 10m/sec in that little bit of push time.

The general formula for calculating the velocity of an accelerating object is:

Velocity = Acceleration * Time

Physics shorthand: v = at

Let's relate this back to our jump. If the landing velocity is the same as the push velocity, we know that:

v = Jump Acceleration * Jump Time

Hence,

Jump Acceleration * Jump Time = Push Acceleration * Push Time

Moving things around with a bit of algebra, we arrive at this equation:

$$\frac{Jump\ Time}{Push\ Time} = \frac{Push\ Acceleration}{Jump\ Acceleration\ (Gravity)}$$

Look, it's the JM! And it's equal to the ratio of the push acceleration to gravity. Increase your jump time, and the push acceleration goes up. Decrease your push time, and the push

acceleration goes up. Distance (or in this case, jump or push height) is also related to velocity:

Distance = Average Velocity * Time

Physics shorthand:

$$d = vt$$

With some algebra, we make this into yet another formula for the average

velocity: $v = d/t$

Because the average velocity is the same for both the push and jump, we can say that d/t is the same for both jump and

push:

Jump Height/Jump Time = Push Height/Push Time

And with a little more algebra:

$$\frac{Jump\ Height}{Push\ Time} = \frac{Push\ Height}{Push\ Time}$$

Push Time

The JM also gives you the ratio of the jump time to the push time.

JM = Jump Time/Push Time

Working a little algebra, we can express the equation in a way that directly calculates the push time:

Push Time = Jump Time/JM

Example:

JM = 3

Jump Time: 15 frames

Push Time = 15/3 = 5 frames

Landing

The forces on landing are similar to takeoff. If the landing has faster timing, the forces will be larger than for a longer timing.

5.1.11.4. Stop Time

The stop height is often a bit larger than the push height, but the timing ofthe push and stop are the same in the sense that the CG moves the same distance per frame in the push and stop.

If the stop height is larger than the push height, you'll just need more frames for the stop than the push.

Push Height/Push Frames = Stop Height/Stop Frames

This can also be expressed as:

Push Height/Push Time = Stop Distance/Stop Time

You can also flip everything over and express it as:

Push Time/Push Height = Stop Time/Stop Distance

Using algebra, we can get the following equation for stop time:

Stop Time = (Push Time * Stop Distance)/Push Height

Example:

Push Time: 5 frames

Push Height: 0.4m

Stop Height: 0.5m

Stop Time = (5 * 0.5) / 0.4 = 6 frames

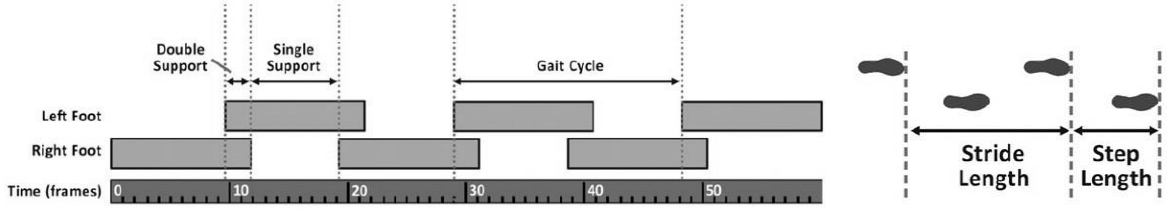
5.1.12. Walking

Walks feature all the basics of mechanics while including personality. The ability to animate walk cycles is one of the most important skills a character animator needs to master.

5.1.12.1. Strides and Steps

A step is one step with one foot. A stride is two steps, one with each foot. Stride length is the distance the character travels in a stride, measured from the same part of the foot. Step and stride length indicate lengthwise spacing for the feet during a walk.

Gait is the timing of the motion for each foot, including how long each foot is on the ground or in the air. During a walk, the number of feet the character has on the ground changes from one foot (single support) to two feet (double support) and then back to one foot. You can plot the time each foot is on the ground to see the single and double support times over time. A normal walking gait ranges from 1/ 3 to 2/3 of a second per step, with 1/2 second being average.



5.1.12.2. Walk Timing: Walking is sometimes called "controlled falling." Right after you move past the passing position, your body's center of gravity is no longer over your base of

support, and you begin to tip. Your passing leg moves forward to stop the fall, creating your next step. Then the cycle begins again. The horizontal timing for between the four walk poses is not uniform. The CG slows in going from the contact to passing position, then slows out from passing to contact. The CG also rises and falls, rising to the highest position during passing and the lowest during contact. The head is in the highest position during passing.

5.2. Statistical Physics for Computing

Note: Basic Definitions

[Mean: The mean is a measure of central tendency that is calculated by adding up all the values in a dataset and dividing by the number of values. It is often referred to as the "average" of the data.

Median: The median is the middle value in a dataset when the values are arranged in order. It is often used as an alternative to the mean when the data has extreme values that can skew the mean.

Mode: The mode is the value that appears most frequently in a dataset. It is often used as an alternative to the mean and median for describing the typical value of a dataset.

Variance: Variance is a measure of how spread out a dataset is. It is calculated by taking the average of the squared differences between each data point and the mean.

Standard deviation: The standard deviation is a measure of the spread of a dataset that is calculated as the square root of the variance. It tells us how much the data deviates from the mean.

Range: The range is the difference between the largest and smallest values in a dataset. It provides a measure of the spread of the data].

5.2.1. Descriptive statistics and inferential statistics:

Descriptive statistics and inferential statistics are two different branches of statistics used for analyzing and interpreting data.

Descriptive statistics are used to summarize and describe the main features of a dataset. This includes measures such as mean, median, mode, variance, standard deviation, and range. Descriptive statistics provide a way to organize and present data in a meaningful and informative way. They help to identify patterns, relationships, and trends within the data.

Inferential statistics, on the other hand, are used to draw conclusions and make predictions about a larger population based on a sample of data. Inferential statistics involve making estimates, testing hypotheses, and making predictions using the sample data. This allows us to generalize the findings from the sample to the larger population. Inferential

statistics are used to make decisions, form policies, and draw conclusions about populations based on sample data.

In summary, descriptive statistics provide a summary of the data and inferential statistics allow us to draw conclusions or make predictions about a larger population based on sample data. Both descriptive and inferential statistics are important tools for analyzing data in many fields, including social sciences, business, and healthcare.

Difference between Descriptive statistics and inferential statistics:

S.No.	Descriptive Statistics	Inferential Statistics
1.	It gives information about raw data which describes the data in some manner.	It makes inferences about the population using data drawn from the population.
2.	It helps in organizing, analyzing, and to present data in a meaningful manner.	It allows us to compare data, and make hypotheses and predictions.
3.	It is used to describe a situation.	It is used to explain the chance of occurrence of an event.
4.	It explains already known data and is limited to a sample or population having a small size.	It attempts to reach the conclusion about the population.
5.	It can be achieved with the help of charts, graphs, tables, etc.	It can be achieved by probability.

5.2.2. Poisson Distribution

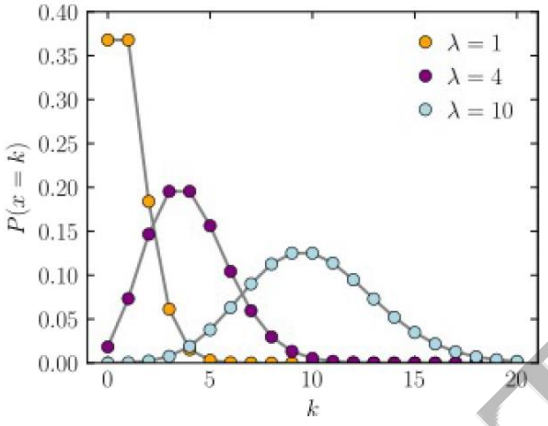
If the probability p is so small that the function has significant value only for very small k, then the distribution of events can be approximated by the Poisson Distribution.

Probability mass function

A discrete Radom variable X is said to have a Poisson distribution, with parameter, if it has a probability Mass Function given by

$$f(k;\lambda) = P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

Here k is the number of occurrences, e is Euler's Number, ! is the factorial function. The positive real number λ is equal to the expected value of X and also to its Variance. The Poisson distribution may be used in the design of experiments such as scattering experiments where a small number of events are seen.



Example of probability for Poisson distributions

On a particular river, overflow floods occur once every 100 years on average. Calculate the probability of k = 0, 1, 2, 3, 4, 5, or 6 overflow floods in a 100 year interval, assuming the Poisson model is appropriate.

Because the average event rate is one overflow flood per 100 years, λ = 1

$$f(k;\lambda) = P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$\text{P(k overflow floods in 100 years)} = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{1^k e^{-1}}{k!}$$

$$\text{P(k} = 0 \text{ overflow floods in 100 years)} = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{1^0 e^{-1}}{0!} = \frac{e^{-1}}{1} = 0.368$$

$$\text{P(k} = 1 \text{ overflow floods in 100 years)} = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{1^1 e^{-1}}{1!} = \frac{e^{-1}}{1} = 0.368$$

$$\text{P(k} = 2 \text{ overflow floods in 100 years)} = \frac{\lambda^k e^{-\lambda}}{k!} = \frac{1^2 e^{-1}}{2!} = \frac{e^{-1}}{2} = 0.184$$

Modeling the Probability for Proton Decay

The experimental search for Proton Decay was undertaken because of the implications of the Grand Unification Theories. The lower bound for the lifetime is now projected to be on the order of $\tau = 10^{33}$ Years. The probability for observing a proton decay can be estimated from the nature of particle decay and the application of Poisson Statistics.

The number of protons N can be modeled by the decay equation

$$N = N_0 e^{-\lambda t}$$

Here $\lambda = 1/ó = 10^{-33}$/ year is the probability that any given proton will decay in a year. Since the decay constant λ is so small, the exponential can be represented by the first two terms of the Exponential Series.

$$e^{-\lambda t} = 1-\lambda t \text{ , } thus\ N \approx N_0 (1-\lambda t)$$

For a small sample, the observation of a proton decay is infinitesmal, but suppose we consider the volume of protons represented by the Super Kameokande neutrino detector in Japan. The number of protons in the detector volume is reported by Ed Kearns of Boston University to be 7.5 x 10^{33} protons. For one year of observation, the number of expected proton decays is then

$$N = N_0 - N_0 \lambda t$$

$$N - N_0 = -N_0 \lambda t$$

$$N_0 - N = N_0 \lambda t$$

$$N_0 - N = (7.5\times10^{33}\ protons)\ (10^{-33}\ /\ year)\ (1\ year) = 7.5$$

About 40% of the area around the detector tank is covered by photo-detector tubes, and if we take that to be the nominal efficiency of detection, we expect about three observations of proton decay events per year based on a 10^{33}year lifetime.

So far, no convincing proton decay events have been seen. Poisson statistics provides a convenient means for assessing the implications of the absence of these observations. If we presume that $\lambda = 3$ observed decays per year is the mean, then the Poisson distribution function tells us that the probability for zero observations of a decay is

$$p(k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad p(k) = \frac{3^0 e^{-3}}{0!} = 0.05$$

This low probability for a null result suggests that the proposed lifetime of 10^{33} years is too short. While this is not a realistic assessment of the probability of observations because there are a number of possible pathways for decay, it serves to illustrate in principle how even a non-observation can be used to refine a proposed lifetime.

5.2.3. Normal Distribution and Bell Curves

In probability theory and statistics, the normal distribution also called the Gaussian distribution and it is most significant continuous probability distribution. Sometimes also called a bell curve.

A normal distribution comes with a perfectly symmetrical shape.This means that the distribution curve can be divided in the middle to produce two equal halves .The symmetric shape occurs when one half of the observations fall on each side of the curve.

The highest point on the curve, or the top of the bell, represents the most probable event in a series of data (its Mean, Mode and Median in this case), while all other possible occurrences are symmetrically distributed around the mean, creating a downward-sloping curve on each side of the peak. The width of the bell curve is described by its Standard Deviation.

Normal distribution formula:

The probability density function of normal or Gaussian distribution is given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

x = variable

μ = Mean

σ =Standard Deviation

Standard Deviations

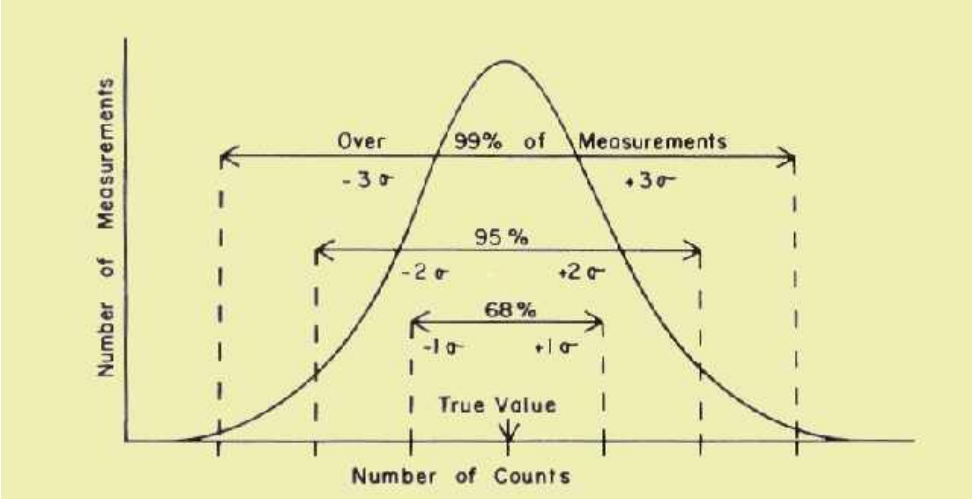
- The width of bell curve is described by its standard deviation.
- The standard deviation is a measurement used to quantify the variability of data dispersion, in a set of given values around the mean.
- The mean refers to all data points in the dataset or sequence and will be found at the highest point on the bell curve.

The Standard Deviation is a measure of how spread outnumbers are,

68% of values are within 1 standard deviation of the mean.

95% of values are within 2 standard deviations of the mean.

99.7% of values are within 3 standard deviations of the mean



Real Life Examples Of Normal Distribution:

- Height. The height of people is an example of normal distribution. ...
- Rolling A Dice. A fair rolling of dice is also a good example of normal distribution. ...
- Tossing A Coin

Normal Distribution Problems:

Question 1: Calculate the probability density function of normal distribution using the following data. x = 3, μ = 4 and σ = 2.

Solution: Given, variable, x = 3

Mean = 4 and

Standard deviation = 2

By the formula of the probability density of normal distribution, we can write;

$$f(3,4,2) = \frac{1}{2\sqrt{2\pi}} e^{\frac{-(3-2)^2}{2\times 2^2}}$$

Hence, f(3,4,2) = 0.176.

5.2.4. Monte-Carlo Method

The Monte Carlo method is a technique that uses random sampling to solve problems. One example of its application is in estimating the value of the mathematical constant π using random numbers.

Monte Carlo methods vary, but tend to follow a particular pattern:

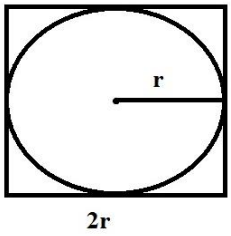
1. Define a domain of possible inputs
2. Generate inputs randomly from a probability distribution over the domain
3. Perform a deterministic computation on the inputs
4. Aggregate the results

Estimating π value using Monte Carlo simulation

Consider a square with sides of length 2r.

Area of square=$S_{square} = (2r)^2 = 4r^2$

Draw the circle inside the square with radius r,



Area of the circle = $S_{circle} = \pi r^2$

Then the ratio is,

$$\frac{S_{circle}}{S_{square}} = \frac{\pi r^2}{4r^2} = \frac{\pi}{4}$$

Multiplying 4 on both sides of equation yields,

$$4 \text{ X } \frac{S_{circle}}{S_{square}} = 4 \text{ X } \frac{\pi}{4}$$

$$4 \text{ X } \frac{S_{circle}}{S_{square}} = \pi$$

$$\Pi = 4 \text{ X } \frac{S_{circle}}{S_{square}}$$

Draw many points randomly selected with in the square,

$$\frac{S_{circle}}{S_{square}} \approx \frac{\text{Points within the circle}}{\text{Points within the square}}$$

Since $\Pi = 4 \text{ X } \frac{S_{circle}}{S_{square}}$

$$\Pi = 4 \text{ X } \frac{S_{circle}}{S_{square}} \approx 4 * \frac{\text{Points within the circle}}{\text{Points within the square}}$$

As we draw more and more points from the square, the approximation will become better and better

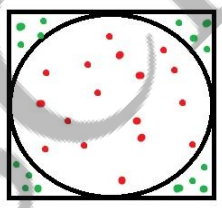
Simulation:

$$\Pi = 4 * \frac{\text{Points within the circle}}{\text{Points within the square}}$$

$\Pi = 4*\frac{2}{4}$.....$4*\frac{583}{741}$.....$4*\frac{664}{845}$

Π=2.000.....3.147.....3.14.....

π = 3.14



There are two important considerations:

1. If the points are not uniformly distributed, then the approximation will be poor.

2. There are many points. The approximation is generally poor if only a few points are randomly placed in the whole square. On average, the approximation improves as more points are placed.

Uses:

1.The monte carlo method simulation is a mathematical technique that predicts possible outcomes of uncertain event.

2. Computer programs use this method to analyze past data and predict a range of future outcomes based on a choice of action.

Problem: 1

1. The number of particles emitted /second by a random radioactive source has a Poisson distribution with λ =4. Calculate the probability of P (X=0), P(X=1) and P(X=2).

Solution: Let X is said to be Poisson distribution

P(X=0) =?

P(X=1) =?

P(X=2) =?

We have **f(k) = P (X=K) =** $\frac{e^{-\lambda}\lambda^{k}}{k!}$

Case (i): P (X=0) i.e. k=0 **λ =4**

P (X=0) = $\frac{e^{-4}4^{0}}{0!}$

P (X=0) = $\frac{0.018}{1}$

P (X=0) = 0.018

Case (ii) : P (X=k) = $\frac{e^{-\lambda}\lambda^{k}}{k!}$

P (X=1) i.e. k=1 $\lambda = 4$

P (X=1) = $\frac{e^{-4}4^{1}}{1!}$

P (X=1) = 0.018 x 4

P (X=1) = 0.0732

Case (ii) : P (X=k) = $\frac{e^{-\lambda}\lambda^{k}}{k!}$

P (X=2) i.e k=2 $\lambda = 4$

P (X=2) = $\frac{e^{-4}4^{1}}{1!}$

P (X=2) = $\frac{0.018 \text{ x } 16}{2!}$

P (X=2) = 0.144

Problem 2: On a Particular place, volcanic eruption occurs once every 100 years on average. Calculate the Probability of k=0, 1, and 2 volcanic eruptions in a 100 years' interval, assuming the Poisson model is appropriate.

Solution:

We have **f(k) = P (X=K) =** $\frac{e^{-\lambda}\lambda^{k}}{k!}$

Given λ = 1 for every 100 years e =2.718

Case:1

Put K=0 λ = 1

f(k) = $\frac{e^{-1}\lambda^{0}}{0!}$

$f(k) = \frac{1}{e}$

$f(k) = \frac{1}{2.718}$

f(k) = 0.368

Case: 2

Put K=1 λ = 1

f (k) = $\frac{e^{-1}\lambda^{1}}{1!}$

$f(k) = \frac{1\,\lambda^{1}}{e}$

$f(k) = \frac{1}{2.178}$ = 0.368

f (k) = 0.368

Case:3

Put K=2 λ = 1

f(k) = $\frac{e^{-1}\lambda^{2}}{2!} = \frac{e^{-1}1^{2}}{2!} = \frac{e^{-1}\lambda^{2}}{2!}$

f(k) = $\frac{1}{2.718X\ 2}$

f(k) = 0.184

Problem 3: Suppose a certain rare disease occurs on average 5 times in a population of 100,000. What is the probability that in a randomly chosen group of 5000 people, there will be no cases of the disease?

Solution:

Let X is the number of cases of the disease in a group of 500 people.

Then X has a Poisson distribution with mean $\lambda = \frac{100,000}{5000} \times 5 = 100$.

We have **f(k) = P (X=K) =** $\frac{e^{-\lambda}\lambda^{k}}{k!}$

The probability of having no cases of the disease is $P(X = 0) = \frac{e^{-\lambda}\lambda^{0}}{0!}$

$= e^{-100} = 3.72 \times 10^{-44}$.

Therefore, the probability of having no cases of the disease in a group of 500 people is very small.

Problem 4: A call center receives an average of 3 calls per minute. What is the probability that in a randomly chosen 2-minute interval, the call center will receive at least 5 calls?

Solution:

Let X is the number of calls the call center receives in a 2-minute interval.

Then X has a Poisson distribution with mean $\lambda = 2 \times 3 = 6$.

The probability of receiving at least 5 calls is

$P(X >= 5) = 1 – P(X < 5) = 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)]$.

We have **f(k) = P (X=K) =** $\frac{e^{-\lambda}\lambda^{k}}{k!}$

Using the Poisson probability formula, we can calculate:

$P(X = 0) = \frac{e^{-6}6^{0}}{0!} = 0.0025$

$P(X = 1) = \frac{e^{-6}6^{1}}{1!} = 0.0149$

$P(X = 2) = \frac{e^{-6}6^{2}}{2!} = 0.0447$

$P(X = 3) = \frac{e^{-6}6^{3}}{3!} = 0.0895$

$P(X = 4) = \frac{e^{-6}6^{4}}{4!} = 0.1342$

Therefore, P(X >= 5) = 1 - (0.0025 + 0.0149 + 0.0447 + 0.0895 + 0.1342) = 0.7142. Therefore, the probability of receiving at least 5 calls in a 2-minute interval is 0.7142.

Problem 5: The number of customers who enter a store per hour follows a Poisson distribution with a mean of 4. What is the probability that exactly 2 customers will enter the store in a given hour?

Solution:

Using the Poisson probability mass function, we have:

We have **f(k) = P (X=K) =** $\frac{e^{-\lambda}\lambda^{k}}{k!}$

P(X = 2) = $\frac{e^{-4}\,4^{2}}{2!}$ = 0.1465

Therefore, the probability that exactly 2 customers will enter the store in a given hour is 0.1465.